# THERE IS NO TAME AUTOMORPHISM OF $\mathbb{C}^3$ WITH MULTIDEGREE $(4,5,6)$

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Abstract. It is known that not each triple $(d_1,d_2,d_3)$ of positive integers is a multidegree of a tame automorphism of $\mathbb{C}^3$. In this paper we show that there is no tame automorphism of $\mathbb{C}^3$ with multidegree $(4,5,6)$. To do this we show that there is no pair of polynomials $P,Q\in\mathbb{C}[x,y,z]$ with certain properties. These properties do not seem particularly restrictive so the non-existence result can be interesting in its own right.

## 1. Introduction

Let $F=(F_1,\ldots,F_n):\mathbb{C}^n\to\mathbb{C}^n$ be any polynomial mapping. By multidegree of $F$, denoted $\operatorname{mdeg} F$, we mean the sequence $(\deg F_1,\ldots,\deg F_n)$. We can also consider the map $\operatorname{mdeg}:\operatorname{End}(\mathbb{C}^n)\to\mathbb{N}^n$, where $\operatorname{End}(\mathbb{C}^n)$ denotes the set of polynomial endomorphisms of $\mathbb{C}^n$. It is trivial that $\operatorname{mdeg}\left(\operatorname{Aut}(\mathbb{C}^1)\right)=\{1\}$, where $\operatorname{Aut}(\mathbb{C}^n)$ denotes the group of polynomial automorphisms of $\mathbb{C}^n$. Let $\operatorname{Tame}(\mathbb{C}^n)$ denote the group of tame automorphisms of $\mathbb{C}^n$. Then, by the Jung [1] and van der Kulk [8] theorem, we have $\operatorname{mdeg}\left(\operatorname{Aut}(\mathbb{C}^2)\right)=\operatorname{mdeg}\left(\operatorname{Tame}(\mathbb{C}^2)\right)=\left\{(d_1,d_2)\in\mathbb{N}^2: d_1|d_2 \text{ or } d_2|d_1\right\}$.

In higher dimensions the propblem is still not well recognized, even in dimension three. The very first result in this direction [3] says that there is no tame automorphism of $\mathbb{C}^3$ with multidegree $(3,4,5)$. In the same paper it was also observed that for all $d_3\geq d_2\geq 2$, $(2,d_2,d_3)\in\operatorname{mdeg}\left(\operatorname{Tame}(\mathbb{C}^3)\right)$. Later [4] it was proven that if $p_2>p_1>2$ are prime numbers, then $(p_1,p_2,d_3)\in\operatorname{mdeg}\left(\operatorname{Tame}(\mathbb{C}^3)\right)$ if and only if $d_3\in p_1\mathbb{N}+p_2\mathbb{N}$ (i.e. $d_3$ is a linear combination of $p_1$ and $p_2$ with coefficients in $\mathbb{N}$). The next step was establishing [5] the equality $\{(3,d_2,d_3):3\leq d_2\leq d_3\}\cap\operatorname{mdeg}\left(\operatorname{Tame}(\mathbb{C}^3)\right)=\{(3,d_2,d_3):3\leq d_2\leq d_3,\ 3|d_2 \text{ or } d_3\in 3\mathbb{N}+d_2\mathbb{N}\}$. A similar equality can be shown for triples of the form $(5,d_2,d_3)$ with $5\leq d_2\leq d_3$. But this is more difficult to prove [6]. In establishing a similar equality for the set $\{(7,d_2,d_3):7\leq d_2\leq d_3\}\cap\operatorname{mdeg}\left(\operatorname{Tame}(\mathbb{C}^3)\right)$ there is one obstruction: the triple $(7,8,12)$.

Notice that $3,5$ and $7$ are prime numbers. For $d_1=4$, the first composite number, we have no description of the entire set $\{(4,d_2,d_3):4\leq d_2\leq d_3\}\cap\operatorname{mdeg}\left(\operatorname{Tame}(\mathbb{C}^3)\right)$, but only some partial information. It is not hard to prove that if $d_3\geq d_2\geq 4$ are even numbers, then $(4,d_2,d_3)\in\operatorname{mdeg}\left(\operatorname{Tame}(\mathbb{C}^3)\right)$. Also we can prove that if $d_3\geq d_2\geq 4$ are odd, then $(4,d_2,d_3)\in\operatorname{mdeg}\left(\operatorname{Tame}(\mathbb{C}^3)\right)$ if and only if $d_3\in 4\mathbb{N}+d_2\mathbb{N}$. But if $d_2$ is odd and $d_3$ is even, or vice versa, we still do not have a

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complete description of the set $\{(4,d_2,d_3) : 4 \le d_2 \le d_3\} \cap \operatorname{mdeg}\left(\operatorname{Tame}\left(\mathbb{C}^3\right)\right)$. The first unknown (up to now) thing has been whether $(4,5,6) \in \operatorname{mdeg}\left(\operatorname{Tame}\left(\mathbb{C}^3\right)\right)$.

The goal of this paper is to prove the following theorem.

**Theorem 1.** *There is no tame automorphism of $\mathbb{C}^3$ with multidegree $(4,5,6)$.*

To do this, we first prove that existence of such a tame automorphism would imply existence of pair of polynomials $P, Q \in \mathbb{C}[x,y,z]$ with certain properties. Next we show that such a pair does not exist. This is the most difficult step. Since these properties do not look very restrictive, the non-existence result can be interesting in its own right. To prove it we develop a method that can be called $H$-reduction.

## 2. The first reduction

In this section we prove that existence of a tame automorphism of $\mathbb{C}^3$ with multidegree $(4,5,6)$ implies existence of a pair of polynomials with some special properties.

**Theorem 2.** *If there exists a tame automorphism $F = (F_1,F_2,F_3)$ of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (4,5,6)$, then there exists a pair of polynomials $P, Q \in \mathbb{C}[x,y,z]$ such that*

$$\begin{aligned} P &= x + P_2 + P_3 + P_4, \qquad P_4 \neq 0, \\ Q &= z + Q_2 + \cdots + Q_6, \qquad Q_6 \neq 0, \end{aligned}$$

*and*

$$\deg\,[P,Q] \le 3,$$

*where $P_i, Q_i$ are homogeneous polynomials of degree $i$.*

Let us recall that for any $f, g \in k[x_1,\ldots,x_n]$ we denote by $[f,g]$ the Poisson bracket of $f$ and $g$, which is the following formal sum:

$$\sum_{1\le i<j\le n}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)[x_i,x_j],$$

where $[x_i,x_j]$ are formal objects satisfying the condition

$$[x_i,x_j] = -[x_j,x_i] \qquad \text{for all } i,j.$$

We also define

$$\deg\,[x_i,x_j] = 2 \qquad \text{for all } i \neq j,$$

$\deg 0 = -\infty$ and

$$\deg\,[f,g] = \max_{1\le i<j\le n} \deg\left\{\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)[x_i,x_j]\right\}.$$

Since $2 - \infty = -\infty$, we have

$$\deg[f,g] = 2 + \max_{1\le i<j\le n} \deg\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right).$$

From the above equality we have

$$\deg\,[f,g] \le \deg f + \deg g. \tag{1}$$

Actually, we can show more:

**Theorem 3.** *If there exists a tame automorphism* $F=(F_1,F_2,F_3)$ *of* $\mathbb{C}^3$ *with* $\operatorname{mdeg} F=(4,5,6)$, *then there exists a pair of polynomials* $P,Q\in\mathbb{C}[x,y,z]$ *such that*

$$\begin{aligned} P &= x+P_2+P_3+P_4, \qquad P_4\neq 0,\\ Q &= z+Q_2+\cdots+Q_6, \qquad Q_6\neq 0, \end{aligned}$$

*and*

$$\deg[P,Q]=3,$$

*where* $P_i,Q_i$ *are homogeneous polynomials of degree* $i$.

2.1. **Some useful results.** Here we collect some useful results from other papers. The first one is the following result of Shestakov and Umirbaev.

**Theorem 4.** *(*[10]*, Theorem 2) Let* $f,g\in\mathbb{C}[x_1,\ldots,x_n]$ *be a* $p$*-reduced pair, and let* $G(x,y)\in\mathbb{C}[x,y]$ *with* $\deg_y G(x,y)=pq+r, 0\le r<p$. *Then*

$$\deg G(f,g)\geq q\left(p\deg g-\deg g-\deg f+\deg[f,g]\right)+r\deg g.$$

The notions of a *-reduced and a $p$-reduced pair, used in the above theorem, are defined as follows.

**Definition 1.** *(*[10]*, Definition 1) A pair* $f,g\in\mathbb{C}[x_1,\ldots,x_n]$ *is called *-reduced if*
*(i)* $f,g$ *are algebraically independent;*
*(ii)* $\overline{f},\overline{g}$ *are algebraically dependent, where* $\overline{h}$ *denotes the highest homogeneous part of* $h$;
*(iii)* $\overline{f}\notin k[\overline{g}]$ *and* $\overline{g}\notin k[\overline{f}]$.

**Definition 2.** *(*[10]*, Definition 1) Let* $f,g\in\mathbb{C}[x_1,\ldots,x_n]$ *be a *-reduced pair with* $\deg f<\deg g$. *Put* $p=\frac{\deg f}{\gcd(\deg f,\deg g)}$. *Then* $f,g$ *is called a* $p$*-reduced pair.*

The estimate from Theorem 4 is true even if the condition (ii) of Definition 1 is not satisfied. We have the following

**Proposition 5.** *(*[6]*, Proposition 2.6) Let* $f,g\in\mathbb{C}[x_1,\ldots,x_n]$ *satisfy conditions (i) and (iii) of Definition 1. Assume that* $\deg f<\deg g$, *put*

$$p=\frac{\deg f}{\gcd\left(\deg f,\deg g\right)},$$

*and let* $G(x,y)\in\mathbb{C}[x,y]$ *with* $\deg_y G(x,y)=pq+r, 0\le r<p$. *Then*

$$\deg G(f,g)\geq q\left(p\deg g-\deg g-\deg f+\deg[f,g]\right)+r\deg g.$$

We will also need the following four results from [6].

**Lemma 6.** *(*[6]*, Lemma 2.7) If* $f,g\in\mathbb{C}[x_1,\ldots,x_n]$ *and* $L\in GL_n(\mathbb{C})$, *then*

$$\deg[L^*(f),L^*(g)]=\deg[f,g],$$

*where* $L^*(h)=h\circ L$ *for any* $h\in\mathbb{C}[x_1,\ldots,x_n]$.

**Lemma 7.** *(*[6]*, Lemma 3.16) For every mapping* $F:\mathbb{C}^n\to\mathbb{C}^n$ *and every* $L\in GL_n(\mathbb{C})$ *we have*

$$\operatorname{mdeg}(F\circ L)=\operatorname{mdeg}F.$$

**Theorem 8.** *([6], Theorem 3.14) Let* $(d_1, d_2, d_3) \neq (1,1,1)$, $d_1 \leq d_2 \leq d_3$ *be a sequence of positive integers. To prove that there is no tame automorphism* $F$ *of* $\mathbb{C}^3$ *with* $\operatorname{mdeg} F = (d_1, d_2, d_3)$ *it is enough to show that a (hypothetical) automorphism* $F$ *of* $\mathbb{C}^3$ *with* $\operatorname{mdeg} F = (d_1, d_2, d_3)$ *admits neither a reduction of type III nor an elementary reduction. Moreover, if we additionally assume that* $\frac{d_3}{d_2} = \frac{3}{2}$ *or* $3 \nmid d_1$, *then it is enough to show that no (hypothetical) automorphism of* $\mathbb{C}^3$ *with multidegree* $(d_1, d_2, d_3)$ *admits an elementary reduction. In both cases we can restrict our attention to automorphisms* $F : \mathbb{C}^3 \to \mathbb{C}^3$ *such that* $F(0,0,0) = (0,0,0)$.

Let us recall that an automorphism $F = (F_1, F_2, F_3)$ admits an *elementary reduction* if there exists a polynomial $g \in \mathbb{C}[x, y]$ and a permutation $\sigma$ of $\{1,2,3\}$ such that $\deg(F_{\sigma(1)} - g(F_{\sigma(2)}, F_{\sigma(3)})) < \deg F_{\sigma(1)}$.

Because the above theorem in [6] has number 3.14, we will refer to it as the $\Pi$-theorem.

**Lemma 9.** *([6], Lemma 3.19) Let* $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ *be such that*

$$f = x_1 + f_2 + \cdots + f_n, \qquad g = x_2 + g_2 + \cdots + g_m,$$

*where* $f_i, g_i$ *are homogeneous forms of degree* $i$. *If* $\deg [f, g] = 2$, *then* $f, g \in \mathbb{C}[x_1, x_2]$.

The last result that we recall here is the following one due to Moh [9].

**Theorem 10.** *(see also [2]) Let* $F : k^2 \to k^2$ *be a Keller map with* $\deg F \leq 101$. *Then* $F$ *is invertible.*

2.2. **The proofs.**

*Proof.* (of Theorem 2) By the $\Pi$-theorem (Theorem 8) it is enough to show that a hypothetical automorphism $F$ of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (4,5,6)$ does not admit an elementary reduction. Moreover, it is enough to show this for automorphisms $F : \mathbb{C}^3 \to \mathbb{C}^3$ such that $F(0,0,0) = (0,0,0)$.

So let us assume that there is an automorphism $F = (F_1, F_2, F_3) : \mathbb{C}^3 \to \mathbb{C}^3$ with $\operatorname{mdeg} F = (4,5,6)$ such that $F$ admits an elementary reduction of the form $(F_1, F_2, F_3 - g(F_1, F_2))$, where $g \in \mathbb{C}[x, y]$. Then

$$\deg g(F_1, F_2) = \deg F_3 = 6, \tag{2}$$

and by Proposition 5,

$$\deg g(F_1, F_2) \geq q(4 \cdot 5 - 5 - 4 + \deg [F_1, F_2]) + 5r, \tag{3}$$

where $\deg_y g(x, y) = 4q + r$, with $0 \leq r < 4$. Since $4 \cdot 5 - 5 - 4 + \deg [F_1, F_2] \geq 11 + \deg [F_1, F_2] > 6$, by (2) and (3) we have $q = 0$. Also by (2) and (3) we have $r < 2$. Thus $g(x, y) = g_0(x) + y g_0(x)$. And, since $4\mathbb{N} \cap (5 + 4\mathbb{N}) = \emptyset$, it follows that

$$6 = \deg g(F_1, F_2) \in 4\mathbb{N} \cup (5 + 4\mathbb{N}),$$

a contradiction.

Now, assume that $F = (F_1, F_2, F_3) : \mathbb{C}^3 \to \mathbb{C}^3$ is an automorphism such that $\operatorname{mdeg} F = (4,5,6)$ and $F$ admits an elementary reduction of the form $(F_1 - g(F_2, F_3), F_2, F_3)$, where $g \in \mathbb{C}[x, y]$. Then

$$\deg g(F_2, F_3) = \deg F_1 = 4, \tag{4}$$

and by Proposition 5,

$$\deg g(F_2, F_3) \geq q(5 \cdot 6 - 6 - 5 + \deg [F_1, F_3]) + 6r, \tag{5}$$

where $\deg_y g(x,y) = 5q + r$, with $0 \le r < 5$. Since $5 \cdot 6 - 6 - 5 + \deg[F_1, F_3] \ge 29 + \deg[F_1, F_3] > 4$, we have $q = r = 0$. This means that $g(x,y) = g(x)$, and so

$$4 = \deg g(F_2, F_3) = \deg g(F_2) \in 5\mathbb{N},$$

a contradiction.

Finally, assume that $F = (F_1, F_2, F_3)$ is an automorphism of $\mathbb{C}^3$ such that $\operatorname{mdeg} F = (4,5,6)$ and $F$ admits an elementary reduction of the form $(F_1, F_2 - g(F_1, F_3), F_3)$, where $g \in \mathbb{C}[x,y]$. By Theorem 8 we can also assume that $F(0,0,0) = (0,0,0)$. We have

$$\deg g(F_1, F_3) = \deg F_2 = 5, \tag{6}$$

and by Proposition 5,

$$\deg g(F_1, F_3) \ge q(p \cdot 6 - 6 - 4 + \deg[F_2, F_3]) + 6r, \tag{7}$$

where $\deg_y g(x,y) = qp + r$, with $0 \le r < p$ and $p = \frac{4}{\gcd(4,6)} = 2$. By (6) and (7) we see that $r = 0$.

Consider the case $\deg[F_1, F_3] > 3$. Then $p \cdot 6 - 6 - 4 + \deg[F_2, F_3] = 2 + \deg[F_2, F_3] > 5$, and by (6) and (7) we see that $q = 0$. Thus in this case, we have $g(x,y) = g(x)$, and so $\deg g(F_1, F_3) = \deg g(F_1) \in 4\mathbb{N}$. This contradicts (6). Thus, $\deg[F_1, F_3] \le 3$.

Let $L$ be the linear part of the automorphism $F$. Since $F(0,0,0) = (0,0,0)$, the linear part of $F \circ L^{-1}$ is the identity map $\mathrm{id}_{\mathbb{C}^3}$. Thus

$$\begin{aligned} F_1 \circ L^{-1} &= x + \text{higher degree summands}, \\ F_3 \circ L^{-1} &= z + \text{higher degree summands}. \end{aligned} \tag{8}$$

By Lemma 6, we have

$$\deg\left[F_1 \circ L^{-1}, F_3 \circ L^{-1}\right] = \deg[F_1, F_3] \le 3,$$

and by Lemma 7 we have $\deg\left(F_1 \circ L^{-1}\right) = 4, \deg\left(F_3 \circ L^{-1}\right) = 6$. Thus we can take $P = F_1 \circ L^{-1}$ and $Q = F_3 \circ L^{-1}$. □

*Proof.* (of Theorem 3) Assume that there is a tame automorphism of $\mathbb{C}^3$ with multidegree $(4,5,6)$. By Theorem 2 there exists a pair of polynomials $P, Q \in \mathbb{C}[x,y,z]$ such that

$$\begin{aligned} P &= x + P_2 + P_3 + P_4, \qquad P_4 \ne 0, \\ Q &= z + Q_2 + \cdots + Q_6, \qquad Q_6 \ne 0, \end{aligned}$$

and

$$\deg[P, Q] \le 3,$$

where $P_i, Q_i$ are homogeneous polynomials of degree $i$.

Since $P$ and $Q$ are algebraically independent (over $\mathbb{C}$), we have $\deg[P, Q] \ge 2$. Assume that $\deg[P, Q] = 2$. Then, by Lemma 9, we have

$$P, Q \in \mathbb{C}[x, z].$$

But $\deg[P, Q] = 2$ means that

$$Jac(P, Q) \in \mathbb{C}^*$$

(of course we consider here $P, Q$ as functions of two variables $x, z$). Then by Theorem 10 the map $(P, Q) : \mathbb{C}^2 \to \mathbb{C}^2$ is an automorphism. But $4 \nmid 6$ contradicting the Jung - van der Kulk theorem (see e.g. [1], [8] or [2]). This shows that $\deg [P, Q] = 3$. □

## 3. $H$-reduction method

In this section we develop the main tool that will be used in the next sections of the paper. We start with the following lemma in which we use the notation

$$\mathbb{C}[x_1, \ldots, x_n]_d = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f \text{ is homogeneous od degree } d\} \cup \{0\}.$$

**Lemma 11.** *Let $H$ be a squarefree, nonconstant homogeneous polynomial and let $P$ be any homogeneous polynomial such that*

$$[H, P] = 0.$$

*Then there exist $a \in \mathbb{C}$ and $k \in \mathbb{N}$ such that*

$$P = aH^k.$$

*Moreover, if $P \in \mathbb{C}[x_1, \ldots, x_n]_d$ and $\deg H \nmid d$, then $P = 0$.*

*Proof.* Since $[H, P] = 0$, it follows that $H$ and $P$ are algebraically dependent and so by Lemma 2 in [?] there exist $a, b \in \mathbb{C}$, $k_1, k_2 \in \mathbb{N}$ and a homogeneous polynomial $h$ such that

$$P = ah^{k_1} \qquad \text{and} \qquad H = bh^{k_2}.$$

Since $H$ is squarefree, we conclude that $k_2 = 1$ and so we can take $h = H$. Thus, in particular, if $a \neq 0$ (i.e. $P \neq 0$), then $\deg P$ is divisible by $\deg H$. □

**Corollary 12.** *Let $H$ be a squarefree, nonconstant homogeneous polynomial and let $P$ be any polynomial such that*

$$[H, P] = 0.$$

*Then $P \in \mathbb{C}[H]$.*

*Proof.* Let $d = \deg P$ and let $P = P_0 + \cdots + P_d$ be the homogeneous decomposition of $P$. Since $[H, P] = 0$, it follows that

$$[H, P_i] = 0 \qquad \text{for } i = 0, \ldots, d. \tag{9}$$

In particular, $[H, P_d] = 0$. Since $P_d \neq 0$ (by definition of $d$), it follows that $d = k \deg H$ for some $k \in \mathbb{N}$. By (9) and Lemma 11 there exist $a_0, \ldots, a_k \in \mathbb{C}$, $a_k \neq 0$ such that

$$P_{l \deg H} = a_l H^l \qquad \text{for } l = 0, \ldots, k$$

and $P_i = 0$ for $i \notin \{0, \deg H, 2 \deg H, \ldots, k \deg H\}$. □

We will also use the following fact that is easy to check.

**Lemma 13.** *Let $P \in \mathbb{C}[x_1, \ldots, x_n]$. For any $Q, R \in \mathbb{C}[x_1, \ldots, x_n]$ and $\alpha, \beta \in \mathbb{C}$ we have*

$$\begin{aligned} [P, QR] &= Q\,[P, R] + R\,[P, Q]\,, \\ [P, \alpha Q + \beta R] &= \alpha\,[P, Q] + \beta\,[P, R]\,, \\ [P, Q] &= -\,[Q, P]\,. \end{aligned}$$

*In other words, the mappings $Q \mapsto [P, Q]$ and $Q \mapsto [Q, P]$ are $\mathbb{C}$-derivations.*

## 4. Non-existence of a special pair of polynomials - preliminary lemma

In this and the next sections our goal is to prove the following theorem.

**Theorem 14.** *There is no pair of polynomials $F, G \in \mathbb{C}[x,y,z]$ such that*

$$\begin{aligned} F &= x + F_2 + F_3 + F_4, \quad F_4 \neq 0, \\ G &= z + G_2 + \cdots + G_6, \quad G_6 \neq 0, \end{aligned}$$

*and*

$$\deg [F,G] \leq 3,$$

*where $F_i, G_i$ are homogeneous polynomials of degree $i$.*

First of all let us notice that the above theorem and Theorem 2 give Theorem 1. Until the end of the paper we assume that

$$F = x + F_2 + F_3 + F_4, \quad F_4 \neq 0,$$

and

$$G = z + G_2 + \cdots + G_6, \quad G_6 \neq 0.$$

The main idea in proving Theorem 14 is to use $H$-reduction to show that smaller $\deg [F,G]$ gives a closer relation between $F$ and $G$. In other words, smaller $\deg [F,G]$ implies smaller flexibility in choosing $F$ and $G$. And finally, small enough $\deg [F,G]$ implies that there is no space for $F$ and $G$. The first step is the following lemma.

**Lemma 15.** *If $\deg [F,G] < 10$, then either*
*(1) there is a squarefree homogeneous polynomial $H$ of degree 2 and $\alpha \in \mathbb{C}^*$ such that*

$$F_4 = H^2, \qquad G_6 = \alpha H^3,$$

*or*
*(2) there is a homogeneous polynomial $h$ of degree 1 and $\alpha \in \mathbb{C}^*$ such that*

$$F_4 = h^4, \qquad G_6 = \alpha h^6.$$

*Proof.* Since $\deg [F,G] < 10$, we have $[F_4, G_6] = 0$, and so $F_4$ and $G_6$ are algebraically dependent. Thus there is a homogeneous polynomial $\widetilde{H}$, $a, \alpha \in \mathbb{C}^*$ and $k_1, k_2 \in \mathbb{N}^*$ such that

$$F_4 = a\widetilde{H}^{k_1}, \qquad G_6 = \alpha \widetilde{H}^{k_2}.$$

Because $\mathbb{C}$ is algebraically closed we can assume that $a = 1$. Since $\gcd(4,6) = 2$, there are two possibilities: $k_1 = 2$, $k_2 = 3$ or $k_1 = 4$, $k_2 = 6$. In the second case we take $h = \widetilde{H}$. And in the first case, $\widetilde{H}$ is either squarefree or not. If it is squarefree we take $H = \widetilde{H}$. And if it is not squarefree then there exist a homogeneous polynomial $h$ of degree 1 and $\gamma \in \mathbb{C}^*$ such that $\widetilde{H} = \gamma h^2$. But since $\mathbb{C}$ is algebraically closed we can assume that $\gamma = 1$. $\square$

## 5. The case of squarefree $H$

Now we consider the situation of Lemma 15(1).

**Lemma 16.** *Let $\deg [F,G] < 9$ and let $\alpha$ and $H$ be as in Lemma 15(1). Then*

$$G_5 = \frac{3}{2} \alpha H F_3.$$

*Proof.* Since $\deg[F,G]<9$, it follows that

$$[F_4,G_5]+[F_3,G_6]=0.$$

By Lemma 15(1),

$$\begin{aligned}
[F_4,G_5]+[F_3,G_6] &= \left[H^2,G_5\right]+\left[F_3,\alpha H^3\right]\\
&= 2H\,[H,G_5]+3\alpha H^2\,[F_3,H]\\
&= 2H\,[H,G_5]-3\alpha H^2\,[H,F_3]\,.
\end{aligned}$$

Since $H$ is a constant for the derivation $P\mapsto[H,P]$, we see that

$$2H\,[H,G_5]-3\alpha H^2\,[H,F_3]=\left[H,2HG_5-3\alpha H^2F_3\right].$$

Thus $\left[H,2HG_5-3\alpha H^2F_3\right]=0$, and since $2HG_5-3\alpha H^2F_3\in\mathbb{C}[x,y,z]_7$, we have, by Lemma 11, $2HG_5-3\alpha H^2F_3=0$. $\square$

**Lemma 17.** *Let* $\deg[F,G]<8$ *and let* $\alpha$ *and* $H$ *be as in Lemma 16. Then there is a homogeneous polynomial* $\widetilde{F}_1$ *of degree* $1$ *and* $b\in\mathbb{C}$ *such that*

$$\begin{aligned}
F_3 &= H\widetilde{F}_1, \qquad G_5=\frac{3}{2}\alpha H^2\widetilde{F}_1,\\
G_4 &= \frac{3}{8}\alpha H\widetilde{F}_1^2+\frac{3}{2}\alpha HF_2+bH^2.
\end{aligned}$$

*Proof.* Since $\deg[F,G]<8$, we have

$$[F_4,G_4]+[F_3,G_5]+[F_2,G_6]=0.$$

By Lemma 16,

$$[F_4,G_4]=\left[H^2,G_4\right]=2H\,[H,G_4]=[H,2HG_4]\,,$$

$$[F_3,G_5]=\left[F_3,\frac{3}{2}\alpha HF_3\right]=\frac{3}{2}\alpha F_3\,[F_3,H]=-\left[H,\frac{3}{4}\alpha F_3^2\right]$$

and

$$[F_2,G_6]=\left[F_2,\alpha H^3\right]=3\alpha H^2\,[F_2,H]=-\left[H,3\alpha H^2F_2\right].$$

Thus $\left[H,2HG_4-\frac{3}{4}\alpha F_3^2-3\alpha H^2F_2\right]=0$, and so there exists $b\in\mathbb{C}$ such that (see Lemma 11)

$$2HG_4-\frac{3}{4}\alpha F_3^2-3\alpha H^2F_2=2bH^3. \tag{10}$$

Since $H|2HG_4-3\alpha H^2F_2$ and $\alpha\neq0$, we conclude that $H|F_3^2$. Since $H$ is squarefree, it follows that $H|F_3$. Thus there exists a homogeneous polynomial $\widetilde{F}_1$ such that $F_3=H\widetilde{F}_1$. Now (10) can be written as follows:

$$2HG_4-\frac{3}{4}\alpha H^2\widetilde{F}_1^2-3\alpha H^2F_2=2bH^3.$$

$\square$

**Lemma 18.** *Let* $\deg[F,G]<7$ *and let* $\alpha,b,H,\widetilde{F}_1$ *be as in Lemma 17. Then*

$$G_3=-\frac{1}{16}\alpha\widetilde{F}_1^3+bH\widetilde{F}_1+\frac{3}{2}\alpha Hx+\frac{3}{4}\alpha\widetilde{F}_1F_2.$$

*Proof.* Since $\deg [F,G] < 7$, we see that

$$[F_4, G_3] + [F_3, G_4] + [F_2, G_5] + [x, G_6] = 0. \tag{11}$$

By Lemma 16,

$$[F_4, G_3] = \left[H^2, G_3\right] = 2H\,[H, G_3] = [H, 2HG_3]\,, \tag{12}$$

and by Lemma 17,

$$\begin{aligned}
[F_3, G_4] &= \left[H\widetilde{F}_1, \frac{3}{8}\alpha H\widetilde{F}_1^2 + \frac{3}{2}\alpha H F_2 + bH^2\right] \\
&= H\left[\widetilde{F}_1, \frac{3}{8}\alpha H\widetilde{F}_1^2 + \frac{3}{2}\alpha H F_2 + bH^2\right] \\
&\quad +\widetilde{F}_1\left[H, \frac{3}{8}\alpha H\widetilde{F}_1^2 + \frac{3}{2}\alpha H F_2 + bH^2\right] \\
&= \frac{3}{8}\alpha H\widetilde{F}_1^2\left[\widetilde{F}_1, H\right] + \frac{3}{2}\alpha H^2\left[\widetilde{F}_1, F_2\right] + \frac{3}{2}\alpha H F_2\left[\widetilde{F}_1, H\right] \\
&\quad +2bH^2\left[\widetilde{F}_1, H\right] + \frac{3}{4}\alpha H\widetilde{F}_1^2\left[H, \widetilde{F}_1\right] + \frac{3}{2}\alpha H\widetilde{F}_1\,[H, F_2]\,,
\end{aligned} \tag{13}$$

$$\begin{aligned}
[F_2, G_5] &= \left[F_2, \frac{3}{2}\alpha H^2\widetilde{F}_1\right] \\
&= \frac{3}{2}\alpha H^2\left[F_2, \widetilde{F}_1\right] + 3\alpha H\widetilde{F}_1\,[F_2, H]\,,
\end{aligned} \tag{14}$$

$$[x, G_6] = \left[x, \alpha H^3\right] = 3\alpha H^2\,[x, H] = \left[H, -3\alpha H^2 x\right]. \tag{15}$$

Notice that:

$$\frac{3}{2}\alpha H^2\left[\widetilde{F}_1, F_2\right] + \frac{3}{2}\alpha H^2\left[F_2, \widetilde{F}_1\right] = 0, \tag{16}$$

$$\frac{3}{8}\alpha H\widetilde{F}_1^2\left[\widetilde{F}_1, H\right] + \frac{3}{4}\alpha H\widetilde{F}_1^2\left[H, \widetilde{F}_1\right] = \frac{3}{8}\alpha H\widetilde{F}_1^2\left[H, \widetilde{F}_1\right] = \left[H, \frac{1}{8}\alpha H\widetilde{F}_1^3\right], \tag{17}$$

$$\begin{aligned}
&\frac{3}{2}\alpha H F_2\left[\widetilde{F}_1, H\right] + \frac{3}{2}\alpha H\widetilde{F}_1\,[H, F_2] + 3\alpha H\widetilde{F}_1\,[F_2, H] \\
&= \frac{3}{2}\alpha H\left(F_2\left[\widetilde{F}_1, H\right] + \widetilde{F}_1\,[F_2, H]\right) = \left[\frac{3}{2}\alpha H\widetilde{F}_1 F_2, H\right],
\end{aligned} \tag{18}$$

By (11)-(18)

$$\left[H, 2HG_3 + \frac{1}{8}\alpha H\widetilde{F}_1^3 - 2bH^2\widetilde{F}_1 - 3\alpha H^2 x - \frac{3}{2}\alpha H\widetilde{F}_1 F_2\right] = 0.$$

Since $2HG_3 + \frac{1}{8}\alpha H\widetilde{F}_1^3 - 2bH^2\widetilde{F}_1 - 3\alpha H^2 x - \frac{3}{2}\alpha H\widetilde{F}_1 F_2 \in \mathbb{C}[x,y,z]_5$, we conclude that (see Lemma 11)

$$2HG_3 + \frac{1}{8}\alpha H\widetilde{F}_1^3 - 2bH^2\widetilde{F}_1 - 3\alpha H^2 x - \frac{3}{2}\alpha H\widetilde{F}_1 F_2 = 0.$$

This gives the formula for $G_3$. $\square$

**Lemma 19.** *Let* $\deg[F,G]<6$ *and let* $\alpha, b, H, \widetilde{F}_1$ *be as in Lemma 18. Then there exist* $c, d \in \mathbb{C}$ *such that*

$$\begin{aligned}
F_2 &= \frac{1}{4}\left(\widetilde{F}_1^2 + dH\right),\\
G_4 &= \frac{3}{4}\alpha H \widetilde{F}_1^2 + \left(\frac{3}{8}\alpha d + b\right) H^2,\\
G_3 &= \frac{1}{8}\alpha \widetilde{F}_1^3 + \left(b + \frac{3}{16}\alpha d\right) H\widetilde{F}_1 + \frac{3}{2}\alpha H x,\\
G_2 &= AH - \frac{1}{4} b\widetilde{F}_1^2 + \frac{3}{4}\alpha x \widetilde{F}_1,
\end{aligned}$$

*where* $A = \frac{3}{128}\alpha d^2 - \frac{1}{4}bd + \frac{1}{2}c$.

*Proof.* Since $\deg[F,G]<6$, we have

$$[F_4, G_2] + [F_3, G_3] + [F_2, G_4] + [x, G_5] = 0. \tag{19}$$

By Lemma 16,

$$[F_4, G_2] = \left[H^2, G_2\right] = 2H\,[H, G_2] = [H, 2HG_2]\,, \tag{20}$$

and by Lemmas 17 and 18,

$$\begin{aligned}
&[F_3, G_3] \\
=\ & \left[H\widetilde{F}_1, -\frac{1}{16}\alpha\widetilde{F}_1^3 + bH\widetilde{F}_1 + \frac{3}{2}\alpha H x + \frac{3}{4}\alpha\widetilde{F}_1 F_2\right]\\
=\ & \left[H\widetilde{F}_1, -\frac{1}{16}\alpha\widetilde{F}_1^3 + \frac{3}{2}\alpha H x + \frac{3}{4}\alpha\widetilde{F}_1 F_2\right]\\
=\ & H\left[\widetilde{F}_1, -\frac{1}{16}\alpha\widetilde{F}_1^3 + \frac{3}{2}\alpha H x + \frac{3}{4}\alpha\widetilde{F}_1 F_2\right]\\
& +\widetilde{F}_1\left[H, -\frac{1}{16}\alpha\widetilde{F}_1^3 + \frac{3}{2}\alpha H x + \frac{3}{4}\alpha\widetilde{F}_1 F_2\right]\\
=\ & \frac{3}{2}\alpha H x\left[\widetilde{F}_1, H\right] + \frac{3}{2}\alpha H^2\left[\widetilde{F}_1, x\right] + \frac{3}{4}\alpha H\widetilde{F}_1\left[\widetilde{F}_1, F_2\right]\\
& -\frac{3}{16}\alpha\widetilde{F}_1^3\left[H, \widetilde{F}_1\right] + \frac{3}{2}\alpha H\widetilde{F}_1\,[H, x] + \frac{3}{4}\alpha\widetilde{F}_1^2\,[H, F_2] + \frac{3}{4}\alpha\widetilde{F}_1 F_2\left[H, \widetilde{F}_1\right],
\end{aligned}\tag{21}$$

$$\begin{aligned}
[F_2, G_4] &= \left[F_2, \frac{3}{8}\alpha H\widetilde{F}_1^2 + \frac{3}{2}HF_2 + bH^2\right]\\
&= \frac{3}{4}\alpha H\widetilde{F}_1\left[F_2, \widetilde{F}_1\right] + \frac{3}{8}\alpha\widetilde{F}_1^2\,[F_2, H]\\
&\quad + \frac{3}{2}\alpha F_2\,[F_2, H] + 2bH\,[F_2, H]\,,
\end{aligned}\tag{22}$$

$$[x, G_5] = \left[x, \frac{3}{2}\alpha H^2\widetilde{F}_1\right] = \frac{3}{2}\alpha H^2\left[x, \widetilde{F}_1\right] + 3\alpha H\widetilde{F}_1\,[x, H]. \tag{23}$$

Notice that:

$$\frac{3}{4}\alpha H\widetilde{F}_1\left[\widetilde{F}_1,F_2\right]+\frac{3}{4}\alpha H\widetilde{F}_1\left[F_2,\widetilde{F}_1\right]=0, \tag{24}$$

$$\frac{3}{2}\alpha H^2\left[\widetilde{F}_1,x\right]+\frac{3}{2}\alpha H^2\left[x,\widetilde{F}_1\right]=0, \tag{25}$$

$$\begin{aligned}(26)\qquad & \frac{3}{4}\alpha\widetilde{F}_1^2\left[H,F_2\right]+\frac{3}{4}\alpha\widetilde{F}_1F_2\left[H,\widetilde{F}_1\right]+\frac{3}{8}\alpha\widetilde{F}_1^2\left[F_2,H\right]\\ = {} & \frac{3}{8}\alpha\left(\widetilde{F}_1^2\left[H,F_2\right]+2\widetilde{F}_1F_2\left[H,\widetilde{F}_1\right]\right)=\left[H,\frac{3}{8}\alpha\widetilde{F}_1^2F_2\right],\end{aligned}$$

$$\begin{aligned}(27)\qquad & \frac{3}{2}\alpha Hx\left[\widetilde{F}_1,H\right]+\frac{3}{2}\alpha H\widetilde{F}_1\left[H,x\right]+3\alpha H\widetilde{F}_1\left[x,H\right]\\ = {} & \frac{3}{2}\alpha H\left(x\left[\widetilde{F}_1,H\right]+\widetilde{F}_1\left[x,H\right]\right)=\left[\frac{3}{2}\alpha Hx\widetilde{F}_1,H\right].\end{aligned}$$

By (19)-(27) we have

$$\left[H,2HG_2-\frac{3}{64}\alpha\widetilde{F}_1^4-\frac{3}{4}\alpha F_2^2+2bHF_2+\frac{3}{8}\alpha\widetilde{F}_1^2F_2-\frac{3}{2}\alpha Hx\widetilde{F}_1\right]=0.$$

Thus there exists $c\in\mathbb{C}$ such that

$$2HG_2-\frac{3}{64}\alpha\widetilde{F}_1^4-\frac{3}{4}\alpha F_2^2+2bHF_2+\frac{3}{8}\alpha\widetilde{F}_1^2F_2-\frac{3}{2}\alpha Hx\widetilde{F}_1=cH^2. \tag{28}$$

Since $H|2HG_2+2bHF_2-\frac{3}{2}\alpha Hx\widetilde{F}_1$, we conclude that

$$H|-\frac{3}{64}\alpha\left(\widetilde{F}_1^4-8\widetilde{F}_1^2F_2+16F_2^2\right)=-\frac{3}{64}\alpha\left(\widetilde{F}_1^2-4F_2\right)^2.$$

Then $H|\widetilde{F}_1^2-4F_2$, because $H$ is squarefree and $\alpha\neq 0$. Thus there is $d\in\mathbb{C}$ such that $\widetilde{F}_1^2-4F_2=-dH$ or equivalently

$$F_2=\frac{1}{4}\left(\widetilde{F}_1^2+dH\right). \tag{29}$$

Using (29) we can rewrite (28) as

$$2HG_2-\frac{3}{64}\alpha d^2H^2+2bHF_2-\frac{3}{2}\alpha Hx\widetilde{F}_1=cH^2.$$

The last equality and (29) give the formula for $G_2$.

Lemma 18 and (29) also give

$$\begin{aligned}G_3 &= -\frac{1}{16}\alpha\widetilde{F}_1^3+bH\widetilde{F}_1+\frac{3}{2}\alpha Hx+\frac{3}{16}\alpha\widetilde{F}_1^3+\frac{3}{16}\alpha d\widetilde{F}_1H\\ &= \frac{1}{8}\alpha\widetilde{F}_1^3+\left(b+\frac{3}{16}\alpha d\right)H\widetilde{F}_1+\frac{3}{2}\alpha Hx\end{aligned}$$

and Lemma 17 and (29) give

$$\begin{aligned}G_4 &= \frac{3}{8}\alpha H\widetilde{F}_1^2+\frac{3}{8}\alpha H\widetilde{F}_1^2+\frac{3}{8}\alpha dH^2+bH^2\\ &= \frac{3}{4}\alpha H\widetilde{F}_1^2+\left(\frac{3}{8}\alpha d+b\right)H^2.\end{aligned}$$

□

**Lemma 20.** *Let* $\deg [F,G] < 5$ *and let* $\alpha, b, c, d, H, \widetilde{F}_1$ *be as in Lemma 19. Then*

$$\begin{aligned} b &= 0, \\ G_4 &= \frac{3}{4}\alpha H \widetilde{F}_1^2 + \frac{3}{8}\alpha d H^2, \\ G_3 &= \frac{1}{8}\alpha \widetilde{F}_1^3 + \frac{3}{16}\alpha d H \widetilde{F}_1 + \frac{3}{2}\alpha H x, \\ G_2 &= AH + \frac{3}{4}\alpha x \widetilde{F}_1, \\ z &= M\widetilde{F}_1 + \frac{3}{16}\alpha d x, \end{aligned}$$

*where* $M = -\frac{3}{256}\alpha d^2 + \frac{1}{4}c$.

*Proof.* Since $\deg [F,G] < 5$, we see that

$$[F_4, z] + [F_3, G_2] + [F_2, G_3] + [x, G_4] = 0. \tag{30}$$

By Lemma 16,

$$[F_4, z] = \left[H^2, z\right] = 2H\,[H,z] = [H, 2Hz]\,, \tag{31}$$

and by Lemmas 17 and 19,

$$\begin{aligned} [F_3, G_2] &= \left[H\widetilde{F}_1, AH - \frac{1}{4}b\widetilde{F}_1^2 + \frac{3}{4}\alpha x \widetilde{F}_1\right] \\ &= H\left[\widetilde{F}_1, AH - \frac{1}{4}b\widetilde{F}_1^2 + \frac{3}{4}\alpha x \widetilde{F}_1\right] \\ &\quad + \widetilde{F}_1\left[H, AH - \frac{1}{4}b\widetilde{F}_1^2 + \frac{3}{4}\alpha x \widetilde{F}_1\right] \\ &= AH\left[\widetilde{F}_1, H\right] + \frac{3}{4}\alpha H \widetilde{F}_1\left[\widetilde{F}_1, x\right] \\ &\quad - \frac{1}{2}b\widetilde{F}_1^2\left[H, \widetilde{F}_1\right] + \frac{3}{4}\alpha x \widetilde{F}_1\left[H, \widetilde{F}_1\right] + \frac{3}{4}\alpha \widetilde{F}_1^2\,[H,x], \end{aligned} \tag{32}$$

$$\begin{aligned} & [F_2, G_3] \\ &= \left[\frac{1}{4}\widetilde{F}_1^2 + \frac{1}{4}dH, \frac{1}{8}\alpha\widetilde{F}_1^3 + \left(b + \frac{3}{16}\alpha d\right)H\widetilde{F}_1 + \frac{3}{2}\alpha H x\right] \\ &= \frac{1}{2}\widetilde{F}_1\left[\widetilde{F}_1, \frac{1}{8}\alpha\widetilde{F}_1^3 + \left(b + \frac{3}{16}\alpha d\right)H\widetilde{F}_1 + \frac{3}{2}\alpha H x\right] \\ &\quad + \frac{1}{4}d\left[H, \frac{1}{8}\alpha\widetilde{F}_1^3 + \left(b + \frac{3}{16}\alpha d\right)H\widetilde{F}_1 + \frac{3}{2}\alpha H x\right] \\ &= \left(\frac{1}{2}b + \frac{3}{32}\alpha d\right)\widetilde{F}_1^2\left[\widetilde{F}_1, H\right] + \frac{3}{4}\alpha H\widetilde{F}_1\left[\widetilde{F}_1, x\right] + \frac{3}{4}\alpha x\widetilde{F}_1\left[\widetilde{F}_1, H\right] \\ &\quad + \frac{3}{32}\alpha d\widetilde{F}_1^2\left[H, \widetilde{F}_1\right] + \left(\frac{1}{4}bd + \frac{3}{64}\alpha d^2\right)H\left[H, \widetilde{F}_1\right] + \frac{3}{8}\alpha d H\,[H,x], \end{aligned} \tag{33}$$

$$
\begin{aligned}
(34)\qquad [x, G_4] &= \left[x, \frac{3}{4}\alpha H\widetilde{F}_1^2 + \left(\frac{3}{8}\alpha d + b\right) H^2\right] \\
&= \underset{\text{−−+−−+−−}}{\frac{3}{4}\alpha\widetilde{F}_1^2\,[x,H]} + \underset{\text{−−−−−−−−−}}{\frac{3}{2}\alpha H\widetilde{F}_1\left[x,\widetilde{F}_1\right]} + \underset{\text{++++++++++++++++}}{\left(\frac{3}{4}\alpha d + 2b\right) H\,[x,H]}\,.
\end{aligned}
$$

Notice that:

$$
(35)\qquad \underset{\text{−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−}}{\frac{3}{4}\alpha H\widetilde{F}_1\left[\widetilde{F}_1, x\right] + \frac{3}{4}\alpha H\widetilde{F}_1\left[\widetilde{F}_1, x\right] + \frac{3}{2}\alpha H\widetilde{F}_1\left[x, \widetilde{F}_1\right]} = 0,
$$

$$
(36)\qquad \underset{\text{======================}}{\frac{3}{4}\alpha x\widetilde{F}_1\left[H, \widetilde{F}_1\right] + \frac{3}{4}\alpha x\widetilde{F}_1\left[\widetilde{F}_1, H\right]} = 0,
$$

$$
(37)\qquad \underset{\text{−−+−−+−−+−−+−−+−−+−−}}{\frac{3}{4}\alpha\widetilde{F}_1^2\,[H,x] + \frac{3}{4}\alpha\widetilde{F}_1^2\,[x,H]} = 0,
$$

$$
\begin{aligned}
(38)\qquad & \underset{\text{++++++++++++++++++++++}}{\frac{3}{8}\alpha dH\,[H,x] + \left(\frac{3}{4}\alpha d + 2b\right) H\,[x,H]} \\
&= -\left[H, \left(\frac{3}{8}\alpha d + 2b\right) Hx\right],
\end{aligned}
$$

$$
\begin{aligned}
(39)\qquad & -\frac{1}{2}b\widetilde{F}_1^2\left[H, \widetilde{F}_1\right] + \underset{\text{\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}}}{\left(\frac{1}{2}b + \frac{3}{32}\alpha d\right)\widetilde{F}_1^2\left[\widetilde{F}_1, H\right] + \frac{3}{32}\alpha d\widetilde{F}_1^2}\left[H, \widetilde{F}_1\right] \\
&= \left[H, -\frac{1}{6}b\widetilde{F}_1^3\right] - \left[H, \left(\frac{1}{6}b + \frac{1}{32}\alpha d\right)\widetilde{F}_1^3\right] + \left[H, \frac{1}{32}\alpha d\widetilde{F}_1^3\right] \\
&= \left[H, -\frac{1}{3}b\widetilde{F}_1^3\right]
\end{aligned}
$$

By (30)-(39) we have

$$
\left[H, 2Hz - AH\widetilde{F}_1 - \frac{1}{3}b\widetilde{F}_1^3 + \left(\frac{1}{4}bd + \frac{3}{64}\alpha d^2\right) H\widetilde{F}_1 - \left(\frac{3}{8}\alpha d + 2b\right) Hx\right] = 0.
$$

Since $2Hz - AH\widetilde{F}_1 - \frac{1}{3}b\widetilde{F}_1^3 + \left(\frac{1}{4}bd + \frac{3}{64}\alpha d^2\right) H\widetilde{F}_1 - \left(\frac{3}{8}\alpha d + 2b\right) Hx \in \mathbb{C}[x,y,z]_3$, we conclude that

$$
(40)\qquad 2Hz - AH\widetilde{F}_1 - \frac{1}{3}b\widetilde{F}_1^3 + \left(\frac{1}{4}bd + \frac{3}{64}\alpha d^2\right) H\widetilde{F}_1 - \left(\frac{3}{8}\alpha d + 2b\right) Hx = 0.
$$

Since $H | 2Hz - AH\widetilde{F}_1 + \left(\frac{1}{4}bd + \frac{3}{64}\alpha d^2\right) H\widetilde{F}_1 - \left(\frac{3}{8}\alpha d + 2b\right) Hx$, we see that $b = 0$ or $H | \widetilde{F}_1^3$. But $H | \widetilde{F}_1^3$ means that $H$ is not squarefree. Thus $b = 0$. So (40) can be rewritten as

$$
2Hz - AH\widetilde{F}_1 + \frac{3}{64}\alpha d^2 H\widetilde{F}_1 - \frac{3}{8}\alpha dHx = 0.
$$

Thus

$$\begin{aligned} z &= \frac{1}{2}A\widetilde{F}_1 - \frac{3}{128}\alpha d^2\widetilde{F}_1 + \frac{3}{16}\alpha dx \\ &= \left(\frac{3}{256}\alpha d^2 + \frac{1}{4}c\right)\widetilde{F}_1 - \frac{3}{128}\alpha d^2\widetilde{F}_1 + \frac{3}{16}\alpha dx \\ &= \left(-\frac{3}{256}\alpha d^2 + \frac{1}{4}c\right)\widetilde{F}_1 + \frac{3}{16}\alpha dx. \end{aligned}$$

The formulas for $G_4, G_3$ and $G_2$ are obtained by substituting $b = 0$ in the formulas from Lemma 19. $\square$

Now we are in a position to prove

**Theorem 21.** *There is no pair of polynomials $F, G$ of the form*

$$\begin{aligned} F &= x + F_2 + F_3 + F_4, \qquad F_4 \neq 0, \\ G &= z + G_2 + \cdots + G_6, \qquad G_6 \neq 0, \end{aligned}$$

*where $F_4, G_6$ are given by the formulas of Lemma 15(1), such that* $\deg [F, G] < 4$.

*Proof.* Assume that there exists such a pair. Then

$$[F_3, z] + [F_2, G_2] + [x, G_3] = 0. \tag{41}$$

By Lemmas 17 and 20,

$$\begin{aligned} [F_3, z] &= \left[H\widetilde{F}_1, M\widetilde{F}_1 + \frac{3}{16}\alpha dx\right] \\ &= H\left[\widetilde{F}_1, M\widetilde{F}_1 + \frac{3}{16}\alpha dx\right] + \widetilde{F}_1\left[H, M\widetilde{F}_1 + \frac{3}{16}\alpha dx\right] \\ &= \frac{3}{16}\alpha dH\left[\widetilde{F}_1, x\right] + M\widetilde{F}_1\left[H, \widetilde{F}_1\right] + \frac{3}{16}\alpha d\widetilde{F}_1\,[H, x], \end{aligned} \tag{42}$$

and by Lemmas 19 and 20,

$$\begin{aligned} [F_2, G_2] &= \left[\frac{1}{4}\widetilde{F}_1^2 + \frac{1}{4}dH, AH + \frac{3}{4}\alpha x\widetilde{F}_1\right] \\ &= \frac{1}{2}\widetilde{F}_1\left[\widetilde{F}_1, AH + \frac{3}{4}\alpha x\widetilde{F}_1\right] \\ &\quad + \frac{1}{4}d\left[H, AH + \frac{3}{4}\alpha x\widetilde{F}_1\right] \\ &= \frac{1}{2}A\widetilde{F}_1\left[\widetilde{F}_1, H\right] + \frac{3}{8}\alpha\widetilde{F}_1^2\left[\widetilde{F}_1, x\right] \\ &\quad + \left[H, \frac{3}{16}\alpha dx\widetilde{F}_1\right], \end{aligned} \tag{43}$$

$$
\begin{aligned}
[x, G_3] &= \left[x, \frac{1}{8}\alpha \widetilde{F}_1^3 + \frac{3}{16}\alpha d H \widetilde{F}_1 + \frac{3}{2}\alpha H x\right] \\
&= \frac{3}{8}\alpha \widetilde{F}_1^2 \left[x, \widetilde{F}_1\right] + \frac{3}{16}\alpha d \widetilde{F}_1 \left[x, H\right] \\
&\quad + \frac{3}{16}\alpha d H \left[x, \widetilde{F}_1\right] + \frac{3}{2}\alpha x \left[x, H\right].
\end{aligned} \tag{44}
$$

Notice that:

$$
\frac{3}{8}\alpha \widetilde{F}_1^2 \left[\widetilde{F}_1, x\right] + \frac{3}{8}\alpha \widetilde{F}_1^2 \left[x, \widetilde{F}_1\right] = 0, \tag{45}
$$

$$
\frac{3}{16}\alpha d \widetilde{F}_1 \left[H, x\right] + \frac{3}{16}\alpha d \widetilde{F}_1 \left[x, H\right] = 0, \tag{46}
$$

$$
\frac{3}{16}\alpha d H \left[\widetilde{F}_1, x\right] + \frac{3}{16}\alpha d H \left[x, \widetilde{F}_1\right] = 0. \tag{47}
$$

By (41)-(47) we have

$$
\left[H, \frac{1}{2}M\widetilde{F}_1^2 - \frac{1}{4}A\widetilde{F}_1^2 + \frac{3}{16}\alpha d x \widetilde{F}_1 - \frac{3}{4}\alpha x^2\right] = 0.
$$

Since $\frac{1}{2}M\widetilde{F}_1^2 - \frac{1}{4}A\widetilde{F}_1^2 + \frac{3}{16}\alpha d x \widetilde{F}_1 - \frac{3}{4}\alpha x^2 \in \mathbb{C}[x,y,z]_2$, there is $e \in \mathbb{C}$ such that

$$
\frac{1}{2}M\widetilde{F}_1^2 - \frac{1}{4}A\widetilde{F}_1^2 + \frac{3}{16}\alpha d x \widetilde{F}_1 - \frac{3}{4}\alpha x^2 = eH. \tag{48}
$$

We have (see Lemmas 19 and 20) $\frac{1}{2}M - \frac{1}{4}A = -\frac{3}{256}\alpha d^2$. Thus (48) can be rewritten as

$$
-\frac{3}{256}\alpha d^2 \widetilde{F}_1^2 + \frac{3}{16}\alpha d x \widetilde{F}_1 - \frac{3}{4}\alpha x^2 = eH
$$

or equivalently

$$
-\frac{3}{4}\alpha \left(\frac{1}{8}d\widetilde{F}_1 - x\right)^2 = eH.
$$

Since $H$ is squarefree and $\alpha \neq 0$, we see that $e = 0$ and $\frac{1}{8}d\widetilde{F}_1 - x = 0$. This means that $d \neq 0$ and $\widetilde{F}_1 = 8d^{-1}x$. This contradicts $z = M\widetilde{F}_1 + \frac{3}{16}\alpha d x$. □

## 6. The case of nonsquarefree $H$

In this section we consider the situation of Lemma 15(2).

**Lemma 22.** *Let* $\deg[F,G] < 9$ *and let* $\alpha$ *and* $h$ *be as in Lemma 15(2). Then there is* $\beta \in \mathbb{C}$ *such that:*

$$
G_5 = \frac{3}{2}\alpha h^2 F_3 + \beta h^5.
$$

*Proof.* Since $\deg[F,G] < 9$, it follows that

$$
[F_4, G_5] + [F_3, G_6] = 0.
$$

By Lemma 15(2),

$$\begin{aligned} [F_4, G_5] + [F_3, G_6] &= \left[h^4, G_5\right] + \left[F_3, \alpha h^6\right] \\ &= 4h^3 [h, G_5] + 6\alpha h^5 [F_3, h] \\ &= \left[h, 4h^3 G_5 - 6\alpha h^5 F_3\right]. \end{aligned}$$

Thus $\left[h, 4h^3G_5 - 6\alpha h^5 F_3\right] = 0$. Since $\deg h = 1$ and $4h^3G_5 - 6\alpha h^5 F_3 \in \mathbb{C}[x, y, z]_7$, we conclude (by Lemma 11) that there exists $\beta \in \mathbb{C}$ such that

$$4h^3 G_5 - 6\alpha h^5 F_3 = 4\beta h^7.$$

This gives the formula for $G_5$. □

**Lemma 23.** *Let* $\deg [F, G] < 8$ *and let* $\alpha, \beta, h$ *be as in Lemma 22. Then there is a homogeneous polynomial* $\widetilde{F}_2$ *of degree* 2 *and* $a \in \mathbb{C}$ *such that:*

$$\begin{aligned} F_3 &= h\widetilde{F}_2, \\ G_5 &= \frac{3}{2}\alpha h^3 \widetilde{F}_2 + \beta h^5, \\ G_4 &= \frac{3}{8}\alpha \widetilde{F}_2^2 + \frac{5}{4}\beta h^2 \widetilde{F}_2 + \frac{3}{2}\alpha h^2 F_2 + \frac{1}{4} a h^4. \end{aligned}$$

*Proof.* Since $\deg [F, G] < 8$, it follows that

$$[F_4, G_4] + [F_3, G_5] + [F_2, G_6] = 0.$$

By Lemmas 15(2) and 22,

$$\begin{aligned} & [F_4, G_4] + [F_3, G_5] + [F_2, G_6] \\ = \ & \left[h^4, G_4\right] + \left[F_3, \frac{3}{2}\alpha h^2 F_3 + \beta h^5\right] + \left[F_2, \alpha h^6\right] \\ = \ & 4h^3 [h, G_4] + 3\alpha h F_3 [F_3, h] + 5\beta h^4 [F_3, h] + 6\alpha h^5 [F_2, h] \\ = \ & \left[h, 4h^3 G_4\right] + \left[\frac{3}{2}\alpha h F_3^2, h\right] + \left[5\beta h^4 F_3, h\right] + \left[6\alpha h^5 F_2, h\right] \\ = \ & \left[h, 4h^3 G_4 - \frac{3}{2}\alpha h F_3^2 - 5\beta h^4 F_3 - 6\alpha h^5 F_2\right]. \end{aligned}$$

Thus $\left[h, 4h^3G_4 - \frac{3}{2}\alpha h F_3^2 - 5\beta h^4 F_3 - 6\alpha h^5 F_2\right] = 0$. By Lemma 11 there is $a \in \mathbb{C}$ such that

$$4h^3 G_4 - \frac{3}{2}\alpha h F_3^2 - 5\beta h^4 F_3 - 6\alpha h^5 F_2 = a h^7. \tag{49}$$

Since $h | 4h^3 G_4 - 5\beta h^4 F_3 - 6\alpha h^5 F_2$ and $\alpha \neq 0$, we see that $h^3 | h F_3^2$. Thus $h | F_3$, and so there is a homogeneous polynomial $\widetilde{F}_2$ of degree 2 such that $F_3 = h\widetilde{F}_2$. By the last equality and (49) we have

$$4h^3 G_4 - \frac{3}{2}\alpha h^3 \widetilde{F}_2^2 - 5\beta h^5 \widetilde{F}_2 - 6\alpha h^5 F_2 = a h^7.$$

This gives the formula for $G_4$. The formula for $G_5$ is obtained by substituting $F_3 = h\widetilde{F}_2$ in the formula from Lemma 22 □

**Lemma 24.** *Let* $\deg[F,G] < 7$ *and let* $\alpha, \beta, a, h, \widetilde{F}_2$ *be as in Lemma 23. Then there is a homogeneous polynomial* $\widetilde{F}_1$ *of degree* 1 *and* $c \in \mathbb{C}$ *such that*

$$\begin{aligned}
F_3 &= h^2\widetilde{F}_1,\\
G_5 &= \frac{3}{2}\alpha h^4\widetilde{F}_1 + \beta h^5,\\
G_4 &= \frac{3}{8}\alpha h^2\widetilde{F}_1^2 + \frac{5}{4}\beta h^3\widetilde{F}_1 + \frac{3}{2}\alpha h^2 F_2 + \frac{1}{4}ah^4,\\
G_3 &= \frac{5}{32}\beta h\widetilde{F}_1^2 + \frac{1}{4}ah^2\widetilde{F}_1 - \frac{1}{16}\alpha\widetilde{F}_1^3 + \frac{5}{4}\beta hF_2 + \frac{3}{2}\alpha h^2 x + \frac{3}{4}\alpha F_2\widetilde{F}_1 + \frac{1}{4}ch^3.
\end{aligned}$$

*Proof.* Since $\deg[F,G] < 7$, it follows that

$$[F_4, G_3] + [F_3, G_4] + [F_2, G_5] + [x, G_6] = 0. \tag{50}$$

By Lemma 15(2),

$$[F_4, G_3] = \left[h^4, G_3\right] = 4h^3\,[h, G_3] = \left[h, 4h^3G_3\right] \tag{51}$$

and

$$[x, G_6] = \left[x, \alpha h^6\right] = 6\alpha h^5\,[x,h] = \left[h, -6\alpha h^5 x\right]. \tag{52}$$

And by Lemma 23,

$$\begin{aligned}
&[F_3, G_4] \\
&= \left[h\widetilde{F}_2, \frac{3}{8}\alpha\widetilde{F}_2^2 + \frac{5}{4}\beta h^2\widetilde{F}_2 + \frac{3}{2}\alpha h^2F_2 + \frac{1}{4}ah^4\right]\\
&= h\left[\widetilde{F}_2, \frac{3}{8}\alpha\widetilde{F}_2^2 + \frac{5}{4}\beta h^2\widetilde{F}_2 + \frac{3}{2}\alpha h^2F_2 + \frac{1}{4}ah^4\right] +\\
&\quad \widetilde{F}_2\left[h, \frac{3}{8}\alpha\widetilde{F}_2^2 + \frac{5}{4}\beta h^2\widetilde{F}_2 + \frac{3}{2}\alpha h^2F_2 + \frac{1}{4}ah^4\right]\\
&= \frac{5}{2}\beta h^2\widetilde{F}_2\left[\widetilde{F}_2, h\right] + 3\alpha h^2F_2\left[\widetilde{F}_2, h\right] + \frac{3}{2}\alpha h^3\left[\widetilde{F}_2, F_2\right] +\\
&\quad ah^4\left[\widetilde{F}_2, h\right] + \frac{3}{4}\alpha\widetilde{F}_2^2\left[h, \widetilde{F}_2\right] + \frac{5}{4}\beta h^2\widetilde{F}_2\left[h, \widetilde{F}_2\right] + \frac{3}{2}\alpha h^2\widetilde{F}_2\,[h, F_2]\\
&= -\left[h, \frac{5}{4}\beta h^2\widetilde{F}_2^2\right] + 3\alpha h^2F_2\left[\widetilde{F}_2, h\right] + \frac{3}{2}\alpha h^3\left[\widetilde{F}_2, F_2\right]\\
&\quad -\left[h, ah^4\widetilde{F}_2\right] + \left[h, \frac{1}{4}\alpha\widetilde{F}_2^3\right] + \left[h, \frac{5}{8}\beta h^2\widetilde{F}_2^2\right] + \frac{3}{2}\alpha h^2\widetilde{F}_2\,[h, F_2],
\end{aligned} \tag{53}$$

and

$$\begin{aligned}
[F_2, G_5] &= \left[F_2, \frac{3}{2}\alpha h^3\widetilde{F}_2 + \beta h^5\right]\\
&= \frac{3}{2}\alpha h^3\left[F_2, \widetilde{F}_2\right] + \frac{9}{2}\alpha h^2\widetilde{F}_2\,[F_2, h] + 5\beta h^4\,[F_2, h]\\
&= \frac{3}{2}\alpha h^3\left[F_2, \widetilde{F}_2\right] + \frac{9}{2}\alpha h^2\widetilde{F}_2\,[F_2, h] - \left[h, 5\beta h^4F_2\right].
\end{aligned} \tag{54}$$

Notice that:

$$\frac{3}{2}\alpha h^3\left[\widetilde{F}_2, F_2\right] + \frac{3}{2}\alpha h^3\left[F_2, \widetilde{F}_2\right] = 0, \tag{55}$$

$$-\left[h, \frac{5}{4}\beta h^2 \widetilde{F}_2^2\right] + \left[h, \frac{5}{8}\beta h^2 \widetilde{F}_2^2\right] = \left[h, -\frac{5}{8}\beta h^2 \widetilde{F}_2^2\right], \tag{56}$$

and

$$\begin{aligned} & 3\alpha h^2 F_2 \left[\widetilde{F}_2, h\right] + \frac{3}{2}\alpha h^2 \widetilde{F}_2 \left[h, F_2\right] + \frac{9}{2}\alpha h^2 \widetilde{F}_2 \left[F_2, h\right] \\ = \; & 3\alpha h^2 \left(F_2 \left[\widetilde{F}_2, h\right] + \widetilde{F}_2 \left[F_2, h\right]\right) = -\left[h, 3\alpha h^2 F_2 \widetilde{F}_2\right]. \end{aligned} \tag{57}$$

By (50)-(57) we have

$$\left[h, 4h^3 G_3 - 6\alpha h^5 x - a h^4 \widetilde{F}_2 + \frac{1}{4}\alpha \widetilde{F}_2^3 - 5\beta h^4 F_2 - \frac{5}{8}\beta h^2 \widetilde{F}_2^2 - 3\alpha h^2 F_2 \widetilde{F}_2\right] = 0.$$

Thus there is $c \in \mathbb{C}$ such that

$$\begin{aligned} & 4h^3 G_3 - 6\alpha h^5 x - a h^4 \widetilde{F}_2 + \frac{1}{4}\alpha \widetilde{F}_2^3 \\ & -5\beta h^4 F_2 - \frac{5}{8}\beta h^2 \widetilde{F}_2^2 - 3\alpha h^2 F_2 \widetilde{F}_2 = c h^6. \end{aligned} \tag{58}$$

Since $h | 4h^3 G_3 - 6\alpha h^5 x - a h^4 \widetilde{F}_2 - 5\beta h^4 F_2 - \frac{5}{8}\beta h^2 \widetilde{F}_2^2 - 3\alpha h^2 F_2 \widetilde{F}_2$ and $\alpha \neq 0$, we see that $h | \widetilde{F}_2$. Thus there is a homogeneous polynomial $\widetilde{F}_1$ of degree 1 such that

$$\widetilde{F}_2 = h \widetilde{F}_1. \tag{59}$$

Now, Lemma 23 and (59) give the formulas for $F_3$, $G_5$ and $G_4$, and (58)-(59) give the formula for $G_3$. $\square$

**Lemma 25.** *Let* $\deg [F, G] < 6$, *and let* $\alpha, \beta, a, c, h, \widetilde{F}_1$ *be as in Lemma 24.*
*(1) If* $\beta = 0$, *then there is a homogeneous polynomial* $\widehat{F}_1$ *of degree* 1 *and* $d \in \mathbb{C}$ *such that*

$$\begin{aligned} F_2 &= \frac{1}{4}\left(\widetilde{F}_1^2 + h\widehat{F}_1\right), \\ G_5 &= \frac{3}{2}\alpha h^4 \widetilde{F}_1, \qquad G_4 = \frac{3}{4}\alpha h^2 \widetilde{F}_1^2 + \frac{3}{8}\alpha h^3 \widehat{F}_1 + \frac{1}{4} a h^4, \\ G_3 &= \frac{1}{8}\alpha \widetilde{F}_1^3 + \frac{1}{4} a h^2 \widetilde{F}_1 + \frac{3}{2}\alpha h^2 x + \frac{3}{16}\alpha h \widetilde{F}_1 \widehat{F}_1 + \frac{1}{4} c h^3, \\ G_2 &= \frac{3}{16} c h \widetilde{F}_1 + \frac{3}{128}\alpha \widehat{F}_1^2 + \frac{1}{16} a \left(\widetilde{F}_1^2 + h\widehat{F}_1\right) + \frac{3}{4}\alpha x \widetilde{F}_1 + \frac{1}{4} d h^2. \end{aligned}$$

*(2) If* $\beta \neq 0$, *then there is a homogeneous polynomial* $\overline{F}_1$ *of degree* 1 *and* $b, d \in \mathbb{C}$ *such that*

$$\begin{aligned} F_2 &= h\overline{F}_1, \qquad \widetilde{F}_1 = bh, \qquad F_3 = bh^3, \\ G_5 &= \left(\frac{3}{2}\alpha b + \beta\right) h^5, \qquad G_4 = E h^4 + \frac{3}{2}\alpha h^3 \overline{F}_1, \\ G_3 &= K h^3 + L h^2 \overline{F}_1 + \frac{3}{2}\alpha h^2 x, \\ G_2 &= A h x + B h^2 + C \overline{F}_1^2 + D h \overline{F}_1, \end{aligned}$$

*where*

$$
\begin{aligned}
A &= \frac{5}{4}\beta + \frac{3}{4}\alpha b, \qquad B = -\frac{5}{128}\beta b^3 + \frac{3}{16}cb + \frac{3}{128}\alpha b^4 + \frac{1}{4}d,\\
C &= \frac{3}{8}\alpha, \qquad D = \frac{1}{4}a + \frac{5}{16}\beta b - \frac{3}{16}\alpha b^2,\\
E &= \frac{3}{8}\alpha b^2 + \frac{5}{4}\beta b + \frac{1}{4}a,\\
K &= \frac{5}{32}\beta b^2 + \frac{1}{4}ab - \frac{1}{16}\alpha b^3 + \frac{1}{4}c, \qquad L = \frac{5}{4}\beta + \frac{3}{4}\alpha b.
\end{aligned}
$$

*Proof.* Since $\deg[F,G] < 6$, we see that

$$[F_4, G_2] + [F_3, G_3] + [F_2, G_4] + [x, G_5] = 0. \tag{60}$$

By Lemma 15(2),

$$[F_4, G_2] = \left[h^4, G_2\right] = 4h^3\,[h, G_2] = \left[h, 4h^3 G_2\right], \tag{61}$$

and by Lemma 24,

$$
\begin{aligned}
[x, G_5] &= \left[x,\, \frac{3}{2}\alpha h^4 \widetilde{F}_1 + \beta h^5\right]\\
&= \underset{\text{*************}}{\frac{3}{2}\alpha h^4\left[x, \widetilde{F}_1\right]} + \underset{=========}{6\alpha h^3 \widetilde{F}_1\,[x,h]} + 5\beta h^4\,[x,h].
\end{aligned} \tag{62}
$$

Also by Lemma 24,

$$
\begin{aligned}
&[F_3, G_3]\\
= &\left[h^2\widetilde{F}_1, \frac{5}{32}\beta h \widetilde{F}_1^2 + \frac{1}{4}ah^2\widetilde{F}_1 - \frac{1}{16}\alpha\widetilde{F}_1^3 + \frac{5}{4}\beta h F_2 + \frac{3}{2}\alpha h^2 x + \frac{3}{4}\alpha F_2\widetilde{F}_1 + \frac{1}{4}ch^3\right]\\
= &\left[h^2\widetilde{F}_1, \frac{5}{32}\beta h \widetilde{F}_1^2 - \frac{1}{16}\alpha\widetilde{F}_1^3 + \frac{5}{4}\beta h F_2 + \frac{3}{2}\alpha h^2 x + \frac{3}{4}\alpha F_2\widetilde{F}_1 + \frac{1}{4}ch^3\right]\\
= &\, h^2\left[\widetilde{F}_1, \frac{5}{32}\beta h \widetilde{F}_1^2 - \frac{1}{16}\alpha\widetilde{F}_1^3 + \frac{5}{4}\beta h F_2 + \frac{3}{2}\alpha h^2 x + \frac{3}{4}\alpha F_2\widetilde{F}_1 + \frac{1}{4}ch^3\right]\\
&+2h\widetilde{F}_1\left[h, \frac{5}{32}\beta h \widetilde{F}_1^2 - \frac{1}{16}\alpha\widetilde{F}_1^3 + \frac{5}{4}\beta h F_2 + \frac{3}{2}\alpha h^2 x + \frac{3}{4}\alpha F_2\widetilde{F}_1 + \frac{1}{4}ch^3\right]\\
= &\underset{\text{--\#--\#--\#--}}{\frac{5}{32}\beta h^2\widetilde{F}_1^2\left[\widetilde{F}_1, h\right]} + \underset{\text{----------}}{\frac{5}{4}\beta h^3\left[\widetilde{F}_1, F_2\right]} + \underset{\text{-*-*-*-*-*-*-}}{\frac{5}{4}\beta h^2 F_2\left[\widetilde{F}_1, h\right]}\\
&+\underset{\text{*************}}{\frac{3}{2}\alpha h^4\left[\widetilde{F}_1, x\right]} + \underset{=========}{3\alpha h^3 x\left[\widetilde{F}_1, h\right]} + \underset{\text{+++++++++++}}{\frac{3}{4}\alpha h^2\widetilde{F}_1\left[\widetilde{F}_1, F_2\right]} + \frac{3}{4}ch^4\left[\widetilde{F}_1, h\right]\\
&+\underset{\text{--\#--\#--\#--}}{\frac{5}{8}\beta h^2\widetilde{F}_1^2\left[h, \widetilde{F}_1\right]} - \frac{3}{8}\alpha h\widetilde{F}_1^3\left[h, \widetilde{F}_1\right] + \underset{\text{-*-*-*-*-*-*-}}{\frac{5}{2}\beta h^2\widetilde{F}_1\,[h, F_2]}\\
&+\underset{=========}{3\alpha h^3\widetilde{F}_1\,[h,x]} + \underset{\text{\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}}}{\frac{3}{2}\alpha h\widetilde{F}_1^2\,[h, F_2]} + \underset{\text{\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}\^{}}}{\frac{3}{2}\alpha h\widetilde{F}_1 F_2\left[h, \widetilde{F}_1\right]}
\end{aligned} \tag{63}
$$

and

$$
\begin{aligned}
(64)\qquad [F_2, G_4] &= \left[F_2, \frac{3}{8}\alpha h^2 \widetilde{F}_1^2 + \frac{5}{4}\beta h^3 \widetilde{F}_1 + \frac{3}{2}\alpha h^2 F_2 + \frac{1}{4} a h^4\right] \\
&= \underset{\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge}{\frac{3}{4}\alpha h \widetilde{F}_1^2 \left[F_2, h\right]} + \underset{++++++++++++}{\frac{3}{4}\alpha h^2 \widetilde{F}_1 \left[F_2, \widetilde{F}_1\right]} + \underset{----------}{\frac{5}{4}\beta h^3 \left[F_2, \widetilde{F}_1\right]} \\
&\quad + \underset{-*-*-*-*-*-*-}{\frac{15}{4}\beta h^2 \widetilde{F}_1 \left[F_2, h\right]} + 3\alpha h F_2 \left[F_2, h\right] + a h^3 \left[F_2, h\right].
\end{aligned}
$$

Notice that:

$$
(65)\qquad \underset{--------------------}{\frac{5}{4}\beta h^3 \left[\widetilde{F}_1, F_2\right] + \frac{5}{4}\beta h^3 \left[F_2, \widetilde{F}_1\right]} = 0,
$$

$$
(66)\qquad \underset{****************************}{\frac{3}{2}\alpha h^4 \left[x, \widetilde{F}_1\right] + \frac{3}{2}\alpha h^4 \left[\widetilde{F}_1, x\right]} = 0,
$$

$$
(67)\qquad \underset{++++++++++++++++++++++++}{\frac{3}{4}\alpha h^2 \widetilde{F}_1 \left[\widetilde{F}_1, F_2\right] + \frac{3}{4}\alpha h^2 \widetilde{F}_1 \left[F_2, \widetilde{F}_1\right]} = 0,
$$

anf that:

$$
\begin{aligned}
(68)\qquad & \underset{--\#--\#--\#--}{\frac{5}{32}\beta h^2 \widetilde{F}_1^2 \left[\widetilde{F}_1, h\right] + \frac{5}{8}\beta h^2 \widetilde{F}_1^2 \left[h, \widetilde{F}_1\right]} \\
&= \frac{15}{32}\beta h^2 \widetilde{F}_1^2 \left[h, \widetilde{F}_1\right] = \left[h, \frac{5}{32}\beta h^2 \widetilde{F}_1^3\right],
\end{aligned}
$$

$$
\begin{aligned}
(69)\qquad & \underset{-*-*-*-*-*-*-*-*-*-*-*-*-*-*-*-*-*-*-*-*-*-}{\frac{5}{4}\beta h^2 F_2 \left[\widetilde{F}_1, h\right] + \frac{5}{2}\beta h^2 \widetilde{F}_1 \left[h, F_2\right] + \frac{15}{4}\beta h^2 \widetilde{F}_1 \left[F_2, h\right]} \\
&= \frac{5}{4}\beta h^2 \left(F_2 \left[\widetilde{F}_1, h\right] + \widetilde{F}_1 \left[F_2, h\right]\right) = -\left[h, \frac{5}{4}\beta h^2 \widetilde{F}_1 F_2\right],
\end{aligned}
$$

$$
\begin{aligned}
(70)\qquad & \underset{=================================}{6\alpha h^3 \widetilde{F}_1 \left[x, h\right] + 3\alpha h^3 x \left[\widetilde{F}_1, h\right] + 3\alpha h^3 \widetilde{F}_1 \left[h, x\right]} \\
&= 3\alpha h^3 \left(\widetilde{F}_1 \left[x, h\right] + x \left[\widetilde{F}_1, h\right]\right) = -\left[h, 3\alpha h^3 x \widetilde{F}_1\right],
\end{aligned}
$$

$$
\begin{aligned}
(71)\qquad & \underset{\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge}{\frac{3}{2}\alpha h \widetilde{F}_1^2 \left[h, F_2\right] + \frac{3}{2}\alpha h \widetilde{F}_1 F_2 \left[h, \widetilde{F}_1\right] + \frac{3}{4}\alpha h \widetilde{F}_1^2 \left[F_2, h\right]} \\
&= \frac{3}{4}\alpha h \left(\widetilde{F}_1^2 \left[h, F_2\right] + 2\widetilde{F}_1 F_2 \left[h, \widetilde{F}_1\right]\right) = \left[h, \frac{3}{4}\alpha h \widetilde{F}_1^2 F_2\right].
\end{aligned}
$$

By (60)-(71) we have

$$
\begin{aligned}
& \left[h, 4h^3 G_2\right] + 5\beta h^4 \left[x, h\right] + \frac{3}{4} c h^4 \left[\widetilde{F}_1, h\right] - \frac{3}{8}\alpha h \widetilde{F}_1^3 \left[h, \widetilde{F}_1\right] + 3\alpha h F_2 \left[F_2, h\right] \\
& + a h^3 \left[F_2, h\right] + \left[h, \frac{5}{32}\beta h^2 \widetilde{F}_1^3\right] - \left[h, \frac{5}{4}\beta h^2 \widetilde{F}_1 F_2\right] - \left[h, 3\alpha h^3 x \widetilde{F}_1\right] + \left[h, \frac{3}{4}\alpha h \widetilde{F}_1^2 F_2\right] \\
= \;& 0
\end{aligned}
$$

or equivalently

$$\begin{aligned}(72)\qquad & \left[h, 4h^3G_2\right] - \left[h, 5\beta h^4 x\right] - \left[h, \frac{3}{4}ch^4\widetilde{F}_1\right] \\ & - \left[h, \frac{3}{32}\alpha h\widetilde{F}_1^4\right] - \left[h, \frac{3}{2}\alpha hF_2^2\right] - \left[h, ah^3F_2\right] \\ & + \left[h, \frac{5}{32}\beta h^2\widetilde{F}_1^3\right] - \left[h, \frac{5}{4}\beta h^2\widetilde{F}_1F_2\right] - \left[h, 3\alpha h^3x\widetilde{F}_1\right] + \left[h, \frac{3}{4}\alpha h\widetilde{F}_1^2F_2\right] \\ = \;& 0.\end{aligned}$$

By the last equality and Lemma 11 there exists $d \in \mathbb{C}$ such that

$$\begin{aligned}(73)\qquad & 4h^3G_2 - 5\beta h^4x - \frac{3}{4}ch^4\widetilde{F}_1 - \frac{3}{32}\alpha h\widetilde{F}_1^4 - \frac{3}{2}\alpha hF_2^2 \\ & -ah^3F_2 + \frac{5}{32}\beta h^2\widetilde{F}_1^3 - \frac{5}{4}\beta h^2\widetilde{F}_1F_2 - 3\alpha h^3x\widetilde{F}_1 + \frac{3}{4}\alpha h\widetilde{F}_1^2F_2 \\ = \;& dh^5.\end{aligned}$$

Since $h^2|4h^3G_2 - 5\beta h^4x - \frac{3}{4}ch^4\widetilde{F}_1 - ah^3F_2 + \frac{5}{32}\beta h^2\widetilde{F}_1^3 - \frac{5}{4}\beta h^2\widetilde{F}_1F_2 - 3\alpha h^3x\widetilde{F}_1$, we see that

$$h^2| - \frac{3}{32}\alpha h\widetilde{F}_1^4 + \frac{3}{4}\alpha h\widetilde{F}_1^2F_2 - \frac{3}{2}\alpha hF_2^2 = -\frac{3}{32}\alpha h\left(\widetilde{F}_1^2 - 4F_2\right)^2.$$

Thus $h|\widetilde{F}_1^2 - 4F_2$, and so there exists a homogeneous polynomial $\widehat{F}_1$ of degree 1 such that

$$(74)\qquad -\widehat{F}_1h = \widetilde{F}_1^2 - 4F_2.$$

Then

$$(75)\qquad F_2 = \frac{1}{4}\left(\widetilde{F}_1^2 + h\widehat{F}_1\right).$$

Using (74)-(75) we can rewrite (73) as

$$\begin{aligned}(76)\qquad & 4h^3G_2 - 5\beta h^4x - \frac{3}{4}ch^4\widetilde{F}_1 - \frac{3}{32}\alpha h^3\widehat{F}_1^2 \\ & -ah^3F_2 + \frac{5}{32}\beta h^2\widetilde{F}_1^3 - \frac{5}{4}\beta h^2\widetilde{F}_1F_2 - 3\alpha h^3x\widetilde{F}_1 \\ = \;& dh^5.\end{aligned}$$

Now, since $h^3|4h^3G_2 - 5\beta h^4x - \frac{3}{4}ch^4\widetilde{F}_1 - \frac{3}{32}\alpha h^3\widehat{F}_1^2 - ah^3F_2 - 3\alpha h^3x\widetilde{F}_1$, we conclude that

$$h^3|\frac{5}{32}\beta h^2\widetilde{F}_1^3 - \frac{5}{4}\beta h^2\widetilde{F}_1F_2 = \frac{5}{32}\beta h^2\widetilde{F}_1\left(\widetilde{F}_1^2 - 8F_2\right).$$

Thus $\beta = 0$ or $h|\widetilde{F}_1\left(\widetilde{F}_1^2 - 8F_2\right)$.

In the first case (i.e. $\beta = 0$), by (74)-(76),

$$G_2 = \frac{3}{16}ch\widetilde{F}_1 + \frac{3}{128}\alpha\widehat{F}_1^2 + \frac{1}{16}a\left(\widetilde{F}_1^2 + h\widehat{F}_1\right) + \frac{3}{4}\alpha x\widetilde{F}_1 + \frac{1}{4}dh^2.$$

The formulas for $G_3, G_4$ and $G_5$ are obtained by substituting (75) and $\beta = 0$ in the formulas from Lemma 24.

In the second case (i.e. $h|\widetilde{F}_1\left(\widetilde{F}_1^2 - 8F_2\right)$ and $\beta \neq 0$) we obtain that $h|\widetilde{F}_1, F_2$. Indeed, if $h|\widetilde{F}_1$, then $h|\frac{1}{4}\left(\widetilde{F}_1^2 + h\widehat{F}_1\right) = F_2$. And, if $h|\left(\widetilde{F}_1^2 - 8F_2\right)$, then $h|\left(\widetilde{F}_1^2 - 4F_2\right) -$

$\left(\widetilde{F}_1^2 - 8F_2\right) = 4F_2$, and so $h|\left(\widetilde{F}_1^2 - 4F_2\right) + 4F_2 = \widetilde{F}_1^2$ and $h|\widetilde{F}_1$. Thus there exists a homogeneous polynomial $\overline{F}_1$ of degree 1 and $b \in \mathbb{C}$ such that

$$F_2 = \overline{F}_1 h, \qquad \widetilde{F}_1 = bh. \tag{77}$$

By (77) and (73) we have

$$\begin{aligned} & 4h^3G_2 - 5\beta h^4 x - \frac{3}{4}cbh^5 - \frac{3}{32}\alpha b^4 h^5 - \frac{3}{2}\alpha h^3 \overline{F}_1^2 \\ & -ah^4\overline{F}_1 + \frac{5}{32}\beta b^3 h^5 - \frac{5}{4}\beta b h^4 \overline{F}_1 - 3\alpha b h^4 x + \frac{3}{4}\alpha b^2 h^4 \overline{F}_1 \\ = \; & dh^5 \end{aligned} \tag{78}$$

or equivalently

$$\begin{aligned} G_2 \;=\; & \left(\frac{5}{4}\beta + \frac{3}{4}\alpha b\right) hx + \left(-\frac{5}{128}\beta b^3 + \frac{3}{16}cb + \frac{3}{128}\alpha b^4 + \frac{1}{4}d\right) h^2 \\ & + \frac{3}{8}\alpha \overline{F}_1^2 + \left(\frac{1}{4}a + \frac{5}{16}\beta b - \frac{3}{16}\alpha b^2\right) h \overline{F}_1 \end{aligned} \tag{79}$$

The formulas for $G_3, G_4, G_5$ and $F_3$ are obtained by substituting (77) in the formulas from Lemma 24. □

Now we consider the situation of Lemma 25(1), and we show that if $\deg [F,G] < 5$, then we do not need consider cases $\beta = 0$ and $\beta \neq 0$ separately.

**Lemma 26.** *Let* $\deg [F,G] < 5$ *and let* $\alpha, \beta, a, c, d, h, \widetilde{F}_1, \widehat{F}_1$ *be as in Lemma 25(1) (in particular* $\beta = 0$*). Then there is* $b \in \mathbb{C}$ *such that for* $\overline{F}_1 = \frac{1}{4}\left(b^2 h + \widehat{F}_1\right)$ *the formulas of Lemma 25(2) holds true (of course with* $\beta = 0$*).*

*Proof.* Since $\deg [F,G] < 5$, we have

$$[F_4, z] + [F_3, G_2] + [F_2, G_3] + [x, G_4] = 0. \tag{80}$$

By Lemma 15(2),

$$[F_4, z] = \left[h^4, z\right] = 4h^3 [h, z] = \left[h, 4h^3 z\right]. \tag{81}$$

By Lemma 24 and Lemma 25(1),

$$
\begin{aligned}
(82)\quad & [F_3, G_2] \\
= & \left[h^2\widetilde{F}_1, \frac{3}{16}ch\widetilde{F}_1 + \frac{3}{128}\alpha\widehat{F}_1^2 + \frac{1}{16}a\widetilde{F}_1^2 + \frac{1}{16}ah\widehat{F}_1 + \frac{3}{4}\alpha x\widetilde{F}_1 + \frac{1}{4}dh^2\right] \\
= & h^2\left[\widetilde{F}_1, \frac{3}{16}ch\widetilde{F}_1 + \frac{3}{128}\alpha\widehat{F}_1^2 + \frac{1}{16}a\widetilde{F}_1^2 + \frac{1}{16}ah\widehat{F}_1 + \frac{3}{4}\alpha x\widetilde{F}_1 + \frac{1}{4}dh^2\right] \\
& +2h\widetilde{F}_1\left[h, \frac{3}{16}ch\widetilde{F}_1 + \frac{3}{128}\alpha\widehat{F}_1^2 + \frac{1}{16}a\widetilde{F}_1^2 + \frac{1}{16}ah\widehat{F}_1 + \frac{3}{4}\alpha x\widetilde{F}_1 + \frac{1}{4}dh^2\right] \\
= & \frac{3}{16}ch^2\widetilde{F}_1\left[\widetilde{F}_1, h\right] + \frac{3}{64}\alpha h^2\widehat{F}_1\left[\widetilde{F}_1, \widehat{F}_1\right] + \frac{1}{16}ah^3\left[\widetilde{F}_1, \widehat{F}_1\right] + \frac{1}{16}ah^2\widehat{F}_1\left[\widetilde{F}_1, h\right] \\
& +\frac{3}{4}\alpha h^2\widetilde{F}_1\left[\widetilde{F}_1, x\right] + \frac{1}{2}dh^3\left[\widetilde{F}_1, h\right] \\
& + \frac{3}{8}ch^2\widetilde{F}_1\left[h, \widetilde{F}_1\right] + \frac{3}{32}\alpha h\widetilde{F}_1\widehat{F}_1\left[h, \widehat{F}_1\right] + \frac{1}{4}ah\widetilde{F}_1^2\left[h, \widetilde{F}_1\right] \\
& +\frac{1}{8}ah^2\widetilde{F}_1\left[h, \widehat{F}_1\right] + \frac{3}{2}\alpha hx\widetilde{F}_1\left[h, \widetilde{F}_1\right] + \frac{3}{2}\alpha h\widetilde{F}_1^2\left[h, x\right].
\end{aligned}
$$

By Lemma 25(1),

$$
\begin{aligned}
(83)\qquad [x, G_4] &= \left[x, \frac{3}{4}\alpha h^2\widetilde{F}_1^2 + \frac{3}{8}\alpha h^3\widehat{F}_1 + \frac{1}{4}ah^4\right] \\
&= \frac{3}{2}\alpha h^2\widetilde{F}_1\left[x, \widetilde{F}_1\right] + \frac{3}{2}\alpha h\widetilde{F}_1^2\left[x, h\right] + \frac{9}{8}\alpha h^2\widehat{F}_1\left[x, h\right] \\
&\quad +\frac{3}{8}\alpha h^3\left[x, \widehat{F}_1\right] + ah^3\left[x, h\right],
\end{aligned}
$$

and

$$
\begin{aligned}
(84)\qquad & [F_2, G_3] \\
= \;& \left[\frac{1}{4}\left(\widetilde{F}_1^2 + h\widehat{F}_1\right), \frac{1}{4}ah^2\widetilde{F}_1 + \frac{1}{8}\alpha\widetilde{F}_1^3 + \frac{3}{2}\alpha h^2 x + \frac{3}{16}\alpha h\widehat{F}_1\widetilde{F}_1 + \frac{1}{4}ch^3\right] \\
= \;& \frac{1}{2}\widetilde{F}_1\left[\widetilde{F}_1, \frac{1}{4}ah^2\widetilde{F}_1 + \frac{1}{8}\alpha\widetilde{F}_1^3 + \frac{3}{2}\alpha h^2 x + \frac{3}{16}\alpha h\widehat{F}_1\widetilde{F}_1 + \frac{1}{4}ch^3\right] \\
& + \frac{1}{4}h\left[\widehat{F}_1, \frac{1}{4}ah^2\widetilde{F}_1 + \frac{1}{8}\alpha\widetilde{F}_1^3 + \frac{3}{2}\alpha h^2 x + \frac{3}{16}\alpha h\widehat{F}_1\widetilde{F}_1 + \frac{1}{4}ch^3\right] \\
& + \frac{1}{4}\widehat{F}_1\left[h, \frac{1}{4}ah^2\widetilde{F}_1 + \frac{1}{8}\alpha\widetilde{F}_1^3 + \frac{3}{2}\alpha h^2 x + \frac{3}{16}\alpha h\widehat{F}_1\widetilde{F}_1 + \frac{1}{4}ch^3\right] \\
= \;& \underset{--\#--\#--\#--}{\frac{1}{4}ah\widetilde{F}_1^2\left[\widetilde{F}_1, h\right]} + \underset{----------}{\frac{3}{4}\alpha h^2\widetilde{F}_1\left[\widetilde{F}_1, x\right]} + \underset{*************}{\frac{3}{2}\alpha hx\widetilde{F}_1\left[\widetilde{F}_1, h\right]} \\
& + \underset{=============}{\frac{3}{32}\alpha\widetilde{F}_1^2 h\left[\widetilde{F}_1, \widehat{F}_1\right]} + \underset{--*--*--*--}{\frac{3}{32}\alpha\widetilde{F}_1^2\widehat{F}_1\left[\widetilde{F}_1, h\right]} + \underset{---\#\#---\#\#---}{\frac{3}{8}ch^2\widetilde{F}_1\left[\widetilde{F}_1, h\right]} \\
& + \underset{==*===*==}{\frac{1}{16}ah^3\left[\widehat{F}_1, \widetilde{F}_1\right]} + \underset{++*+++*+++}{\frac{1}{8}ah^2\widetilde{F}_1\left[\widehat{F}_1, h\right]} + \underset{===========}{\frac{3}{32}\alpha h\widetilde{F}_1^2\left[\widehat{F}_1, \widetilde{F}_1\right]} \\
& + \underset{\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge}{\frac{3}{8}\alpha h^3\left[\widehat{F}_1, x\right]} + \underset{+++++++++}{\frac{3}{4}\alpha h^2 x\left[\widehat{F}_1, h\right]} + \underset{--**--**--**--}{\frac{3}{64}\alpha h^2\widehat{F}_1\left[\widehat{F}_1, \widetilde{F}_1\right]} \\
& + \underset{\wedge\wedge--\wedge\wedge--\wedge\wedge--\wedge\wedge}{\frac{3}{64}\alpha h\widehat{F}_1\widetilde{F}_1\left[\widehat{F}_1, h\right]} + \frac{3}{16}ch^3\left[\widehat{F}_1, h\right] \\
& + \underset{\wedge\wedge**\wedge\wedge**\wedge\wedge**\wedge\wedge}{\frac{1}{16}ah^2\widehat{F}_1\left[h, \widetilde{F}_1\right]} + \underset{--*--*--*--}{\frac{3}{32}\alpha\widehat{F}_1\widetilde{F}_1^2\left[h, \widetilde{F}_1\right]} + \underset{+++++++++}{\frac{3}{8}\alpha h^2\widehat{F}_1[h, x]} \\
& + \underset{--+++--+++--}{\frac{3}{64}\alpha h\widehat{F}_1^2\left[h, \widetilde{F}_1\right]} + \underset{\wedge\wedge--\wedge\wedge--\wedge\wedge--\wedge\wedge}{\frac{3}{64}\alpha h\widehat{F}_1\widetilde{F}_1\left[h, \widehat{F}_1\right]}.
\end{aligned}
$$

Notice that:

$$\underset{-------------------------------------}{\frac{3}{4}\alpha h^2\widetilde{F}_1\left[\widetilde{F}_1, x\right] + \frac{3}{2}\alpha h^2\widetilde{F}_1\left[x, \widetilde{F}_1\right] + \frac{3}{4}\alpha h^2\widetilde{F}_1\left[\widetilde{F}_1, x\right]} = 0, \tag{85}$$

$$\underset{--+--+--+--+--+--+--}{\frac{3}{2}\alpha h\widetilde{F}_1^2[h, x] + \frac{3}{2}\alpha h\widetilde{F}_1^2[x, h]} = 0, \tag{86}$$

$$\underset{\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge\wedge}{\frac{3}{8}\alpha h^3\left[x, \widehat{F}_1\right] + \frac{3}{8}\alpha h^3\left[\widehat{F}_1, x\right]} = 0, \tag{87}$$

$$\underset{****************************}{\frac{3}{2}\alpha hx\widetilde{F}_1\left[h, \widetilde{F}_1\right] + \frac{3}{2}\alpha hx\widetilde{F}_1\left[\widetilde{F}_1, h\right]} = 0, \tag{88}$$

$$\underset{=====================}{\frac{3}{32}\alpha\widetilde{F}_1^2 h\left[\widetilde{F}_1, \widehat{F}_1\right] + \frac{3}{32}\alpha h\widetilde{F}_1^2\left[\widehat{F}_1, \widetilde{F}_1\right]} = 0, \tag{89}$$

$$\frac{3}{32}\alpha\widetilde{F}_1^2\widehat{F}_1\left[\widetilde{F}_1,h\right]+\frac{3}{32}\alpha\widehat{F}_1\widetilde{F}_1^2\left[h,\widetilde{F}_1\right]=0, \tag{90}$$

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$$\frac{1}{16}ah^3\left[\widetilde{F}_1,\widehat{F}_1\right]+\frac{1}{16}ah^3\left[\widehat{F}_1,\widetilde{F}_1\right]=0, \tag{91}$$

==∗=====∗=====∗====∗==

$$\frac{1}{8}ah^2\widetilde{F}_1\left[h,\widehat{F}_1\right]+\frac{1}{8}ah^2\widetilde{F}_1\left[\widehat{F}_1,h\right]=0, \tag{92}$$

++∗++∗++∗++∗++∗++∗++∗++

$$\frac{3}{64}\alpha h^2\widehat{F}_1\left[\widetilde{F}_1,\widehat{F}_1\right]+\frac{3}{64}\alpha h^2\widehat{F}_1\left[\widehat{F}_1,\widetilde{F}_1\right]=0, \tag{93}$$

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$$\frac{3}{64}\alpha h\widehat{F}_1\widetilde{F}_1\left[\widehat{F}_1,h\right]+\frac{3}{64}\alpha h\widehat{F}_1\widetilde{F}_1\left[h,\widehat{F}_1\right]=0, \tag{94}$$

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$$\frac{1}{16}ah^2\widehat{F}_1\left[\widetilde{F}_1,h\right]+\frac{1}{16}ah^2\widehat{F}_1\left[h,\widetilde{F}_1\right]=0, \tag{95}$$

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$$\frac{1}{4}ah\widetilde{F}_1^2\left[h,\widetilde{F}_1\right]+\frac{1}{4}ah\widetilde{F}_1^2\left[\widetilde{F}_1,h\right]=0, \tag{96}$$

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$$\frac{3}{8}ch^2\widetilde{F}_1\left[h,\widetilde{F}_1\right]+\frac{3}{8}ch^2\widetilde{F}_1\left[\widetilde{F}_1,h\right]=0 \tag{97}$$

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and that:

$$\begin{aligned} &\frac{9}{8}\alpha h^2\widehat{F}_1[x,h]+\frac{3}{4}\alpha h^2x\left[\widehat{F}_1,h\right]+\frac{3}{8}\alpha h^2\widehat{F}_1[h,x] \\ = &\frac{3}{4}\alpha h^2\left(\widehat{F}_1[x,h]+x\left[\widehat{F}_1,h\right]\right)=-\left[h,\frac{3}{4}\alpha h^2\widehat{F}_1x\right], \end{aligned} \tag{98}$$

++++++++++++++++++++++++++++

$$\begin{aligned} &\frac{3}{32}\alpha h\widetilde{F}_1\widehat{F}_1\left[h,\widehat{F}_1\right]+\frac{3}{64}\alpha h\widehat{F}_1^2\left[h,\widetilde{F}_1\right] \\ = &\frac{3}{64}\alpha h\left(\widehat{F}_1^2\left[h,\widetilde{F}_1\right]+2\widetilde{F}_1\widehat{F}_1\left[h,\widehat{F}_1\right]\right)=\left[h,\frac{3}{64}\alpha h\widetilde{F}_1\widehat{F}_1^2\right]. \end{aligned} \tag{99}$$

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By (80)-(99),

$$\begin{aligned} &\left[h,4h^3z\right]+\frac{3}{16}ch^2\widetilde{F}_1\left[\widetilde{F}_1,h\right]+\frac{1}{2}dh^3\left[\widetilde{F}_1,h\right]+ah^3[x,h] \\ &+\frac{3}{16}ch^3\left[\widehat{F}_1,h\right]-\left[h,\frac{3}{4}\alpha h^2x\widehat{F}_1\right]+\left[h,\frac{3}{64}\alpha h\widetilde{F}_1\widehat{F}_1^2\right] \\ = &\ 0 \end{aligned}$$

or equivalently

$$\begin{aligned}(100)\qquad & \left[h,4h^3z\right]-\left[h,ah^3x\right]-\left[h,\frac{3}{16}ch^3\widehat{F}_1\right]-\left[h,\frac{3}{32}ch^2\widetilde{F}_1^2\right]\\ & -\left[h,\frac{1}{2}dh^3\widetilde{F}_1\right]-\left[h,\frac{3}{4}\alpha h^2x\widehat{F}_1\right]+\left[h,\frac{3}{64}\alpha h\widetilde{F}_1\widehat{F}_1^2\right]\\ = {} & 0.\end{aligned}$$

By Lemma 11 there exists $g\in\mathbb{C}$ such that

$$(101)\quad 4h^3z-\frac{3}{32}ch^2\widetilde{F}_1^2-\frac{1}{2}dh^3\widetilde{F}_1-ah^3x-\frac{3}{16}ch^3\widehat{F}_1-\frac{3}{4}\alpha h^2x\widehat{F}_1+\frac{3}{64}\alpha h\widetilde{F}_1\widehat{F}_1^2=gh^4.$$

Since $h^2|4h^3z-\frac{3}{32}ch^2\widetilde{F}_1^2-\frac{1}{2}dh^3\widetilde{F}_1-ah^3x-\frac{3}{16}ch^3\widehat{F}_1-\frac{3}{4}\alpha h^2x\widehat{F}_1$ and $\alpha\neq 0$, we see that $h^2|h\widetilde{F}_1\widehat{F}_1^2$. Then $h|\widetilde{F}_1$ or $h|\widehat{F}_1$. Notice that in both cases $h|\widetilde{F}_1$. Indeed, if $h|\widehat{F}_1$ then

$$(102)\qquad h^3|4h^3z-\frac{1}{2}dh^3\widetilde{F}_1-ah^3x-\frac{3}{16}ch^3\widehat{F}_1-\frac{3}{4}\alpha h^2x\widehat{F}_1+\frac{3}{64}\alpha h\widetilde{F}_1\widehat{F}_1^2$$

Then, by (101) and (102) $h^3|-\frac{3}{32}ch^2\widetilde{F}_1^2$ and so $h|\widetilde{F}_1$. Thus there is $b\in\mathbb{C}$ such that

$$(103)\qquad \widetilde{F}_1=bh.$$

Now, by (103) and by Lemma 24, $F_3=bh^3$.

And, by (103) and by Lemma 25(1),

$$F_2=\frac{1}{4}\left(b^2h+\widehat{F}_1\right)h.$$

Thus

$$(104)\qquad F_2=h\overline{F}_1\qquad\text{and}\qquad \widehat{F}_1=4\overline{F}_1-b^2h.$$

Now, one can repeat the same arguments as in the last part of the proof of Lemma 25 obtaining the same formulas as in Lemma 25(2) (of course with $\beta=0$). □

Because of Lemma 26, it does not make a difference whether $\beta=0$ or not. Thus in the following lemma we does not assume anything about $\beta$.

**Lemma 27.** *Let* $\deg[F,G]<5$ *and let* $\alpha,a,b,c,d,h,A,B,C,DE,K,L,\overline{F}_1$ *be as in Lemma 25(2) with arbitrary* $\beta$ *(see Lemma 26). Then either*
*(1) there exist* $R,S,M,N,\widetilde{K},\widetilde{L},\widetilde{B},\widetilde{D}\in\mathbb{C}$ *such that*

$$\begin{aligned}x &= R\overline{F}_1+Sh, \qquad z=M\overline{F}_1+Nh,\\ G_3 &= \widetilde{K}h^3+\widetilde{L}h^2\overline{F}_1, \qquad G_2=\widetilde{B}h^2+C\overline{F}_1^2+\widetilde{D}h\overline{F}_1,\end{aligned}$$

*or*
*(2) there exists* $f\in\mathbb{C}$ *such that*

$$\begin{aligned}\overline{F}_1 &= fh, \qquad F_2=fh^2,\\ G_4 &= \left(E+\frac{3}{2}\alpha f\right)h^4, \qquad G_3=(K+Lf)\,h^3+\frac{3}{2}\alpha h^2x,\\ G_2 &= Ahx+\left(B+Cf^2+Df\right)h^2\end{aligned}$$

*and*

$$h=\frac{1}{M}\left[z-\left(E-\frac{3}{4}bA+\frac{3}{4}\alpha f\right)x\right],$$

*where*

$$M = \left(\frac{3}{4}K - \frac{3}{4}bD\right) f + \frac{1}{4}e + \frac{1}{4}\left(3bC - \frac{1}{2}L\right) f^2.$$

*Proof.* Since $\deg [F, G] < 5$, we have

$$[F_4, z] + [F_3, G_2] + [F_2, G_3] + [x, G_4] = 0. \tag{105}$$

By Lemma 15(2),

$$[F_4, z] = \left[h^4, z\right] = 4h^3 [h, z] = \left[h, 4h^3 z\right] \tag{106}$$

and by Lemmas 25(2) and 26,

$$[x, G_4] = \left[x, Eh^4 + \frac{3}{2}\alpha h^3 \overline{F}_1\right] = 4Eh^3 [x, h] + \underset{--------}{\frac{9}{2}\alpha h^2 \overline{F}_1 [x, h]} + \underset{+++++++++}{\frac{3}{2}\alpha h^3 \left[x, \overline{F}_1\right]}, \tag{107}$$

$$\begin{aligned} [F_3, G_2] &= \left[bh^3, Ahx + Bh^2 + C\overline{F}_1^2 + Dh\overline{F}_1\right] \\ &= 3bh^2 \left[h, Ahx + Bh^2 + C\overline{F}_1^2 + Dh\overline{F}_1\right] \\ &= \left[h, 3bAh^3 x + 3bCh^2\overline{F}_1^2 + 3bDh^3\overline{F}_1\right], \end{aligned} \tag{108}$$

and

$$\begin{aligned} & [F_2, G_3] \\ = {} & \left[h\overline{F}_1, Kh^3 + Lh^2\overline{F}_1 + \frac{3}{2}\alpha h^2 x\right] \\ = {} & h\left[\overline{F}_1, Kh^3 + Lh^2\overline{F}_1 + \frac{3}{2}\alpha h^2 x\right] + \overline{F}_1 \left[h, Kh^3 + Lh^2\overline{F}_1 + \frac{3}{2}\alpha h^2 x\right] \\ = {} & 3Kh^3 \left[\overline{F}_1, h\right] + 2Lh^2\overline{F}_1 \left[\overline{F}_1, h\right] + \underset{---------}{3\alpha h^2 x \left[\overline{F}_1, h\right]} + \underset{+++++++++}{\frac{3}{2}\alpha h^3 \left[\overline{F}_1, x\right]} \\ & + Lh^2\overline{F}_1 \left[h, \overline{F}_1\right] + \underset{--------}{\frac{3}{2}\alpha h^2 \overline{F}_1 [h, x]}. \end{aligned} \tag{109}$$

Notice that:

$$\begin{aligned} & \underset{------------------------------}{\frac{9}{2}\alpha h^2 \overline{F}_1 [x, h] + 3\alpha h^2 x \left[\overline{F}_1, h\right] + \frac{3}{2}\alpha h^2 \overline{F}_1 [h, x]} \\ = {} & 3\alpha h^2 \left(\overline{F}_1 [x, h] + x\left[\overline{F}_1, h\right]\right) = 3\alpha h^2 \left[x\overline{F}_1, h\right] = \left[h, -3\alpha h^2 x\overline{F}_1\right], \end{aligned} \tag{110}$$

$$\underset{++++++++++++++++++}{\frac{3}{2}\alpha h^3 \left[x, \overline{F}_1\right] + \frac{3}{2}\alpha h^3 \left[\overline{F}_1, x\right]} = 0. \tag{111}$$

By (105)-(111)

$$\begin{aligned} & \left[h, 4h^3 z\right] - \left[h, 4Eh^3 x\right] + \left[h, 3bAh^3 x + 3bCh^2\overline{F}_1^2 + 3bDh^3\overline{F}_1\right] \\ & - \left[h, 3Kh^3\overline{F}_1\right] - \left[h, Lh^2\overline{F}_1^2\right] + \left[h, \frac{1}{2}Lh^2\overline{F}_1^2\right] + \left[h, -3\alpha h^2 x\overline{F}_1\right] \\ = {} & 0 \end{aligned} \tag{112}$$

By the last equality and Lemma 11 there exists $e \in \mathbb{C}$ such that
(113)
$$4h^3z - 4Eh^3x + 3bAh^3x + 3bCh^2\overline{F}_1^2 + 3bDh^3\overline{F}_1 - 3Kh^3\overline{F}_1 - \frac{1}{2}Lh^2\overline{F}_1^2 - 3\alpha h^2x\overline{F}_1 = eh^4.$$

Since $h^3 | 4h^3z - 4Eh^3x + 3bAh^3x + 3bDh^3\overline{F}_1 - 3Kh^3\overline{F}_1$, we see that $h^3 | \left(3bC - \frac{1}{2}L\right) h^2\overline{F}_1^2 - 3\alpha h^2x\overline{F}_1 = h^2\overline{F}_1 \left[\left(3bC - \frac{1}{2}L\right)\overline{F}_1 - 3\alpha x\right]$. Thus $h | \left(3bC - \frac{1}{2}L\right)\overline{F}_1 - 3\alpha x$ or $h | \overline{F}_1$.

In the first case (i.e. $h | \left(3bC - \frac{1}{2}L\right)\overline{F}_1 - 3\alpha x$) there exists $\gamma \in \mathbb{C}$ such that

$$\left(3bC - \frac{1}{2}L\right)\overline{F}_1 - 3\alpha x = \gamma h \tag{114}$$

or equivalently

$$\begin{aligned} x &= \frac{1}{3\alpha}\left[\left(3bC - \frac{1}{2}L\right)\overline{F}_1 - \gamma h\right] \\ &= \left(\frac{1}{4}b - \frac{5}{24}\frac{\beta}{\alpha}\right)\overline{F}_1 - \frac{\gamma}{3\alpha}h. \end{aligned} \tag{115}$$

This gives the formula for $x$. Using (114) we can rewrite (113) as

$$4h^3z - 4Eh^3x + 3bAh^3x + 3bDh^3\overline{F}_1 - 3Kh^3\overline{F}_1 + \gamma h^3\overline{F}_1 = eh^4.$$

Thus

$$z = (E - 3bA)\,x + \left(\frac{3}{4}K - \frac{3}{4}bD - \frac{1}{4}\gamma\right)\overline{F}_1 + \frac{1}{4}eh. \tag{116}$$

Now, the formula for $z$ is obtained by substituting (115) in (116), and the formulas for $G_3$ and $G_2$ are obtained by substituting (115) in the formulas of Lemma 25(2) or 26.

In the second case (i.e. $h | \overline{F}_1$) there exists $f \in \mathbb{C}$ such that

$$\overline{F}_1 = fh. \tag{117}$$

Then

$$F_2 = fh^2. \tag{118}$$

Usimg (117) we can rewrite (113) as

$$4h^3z - 4Eh^3x + 3bAh^3x + 3bf^2Ch^4 + 3bfDh^4 - 3fKh^4 - \frac{1}{2}f^2Lh^4 - 3\alpha fh^3x = eh^4.$$

Thus

$$z = \left(E - \frac{3}{4}bA + \frac{3}{4}\alpha f\right)x + Mh. \tag{119}$$

where

$$M = \left(\frac{3}{4}K - \frac{3}{4}bD\right)f + \frac{1}{4}e + \frac{1}{4}\left(3bC - \frac{1}{2}L\right)f^2.$$

By (119) $M \neq 0$ and $h = \frac{1}{M}\left[z - \left(E - \frac{3}{4}bA + \frac{3}{4}\alpha f\right)x\right]$. Substituting $\overline{F}_1 = fh$ in the formulas of Lemma 25(2) gives the formulas for $G_2, G_3$ and $G_4$. $\square$

Now we are in a position to prove

**Theorem 28.** *There is no pair of polynomials $F, G$ of the form*

$$\begin{aligned} F &= x + F_2 + F_3 + F_4, \qquad F_4 \neq 0,\\ G &= z + G_2 + \cdots + G_6, \qquad G_6 \neq 0, \end{aligned}$$

*where $F_4, G_6$ are given by the formulas of Lemma 15(2), such that* $\deg [F,G] < 4$.

*Proof.* Assume that there exists such a pair. Then

$$[F_3, z] + [F_2, G_2] + [x, G_3] = 0. \tag{120}$$

Assume that $F$ and $G$ satisfy Lemma 27(2). Then

$$[F_3, z] = \left[bh^3, z\right] = 3bh^2 [h, z] = \left[h, 3bh^2 z\right],$$

$$\begin{aligned} [F_2, G_2] &= \left[fh^2, Ahx + \left(B + Cf^2 + Df\right) h^2\right]\\ &= 2fh\,[h, Ahx] = \left[h, 2fAh^2x\right] \end{aligned}$$

and

$$[x, G_3] = \left[x, Ph^3 + \frac{3}{2}\alpha h^2 x\right],$$

where $P = K + Lf$. Thus

$$[x, G_3] = 3Ph^2 [x,h] + 3\alpha hx\,[x,h] = -\left[h, 3Ph^2x\right] - \left[h, \frac{3}{2}\alpha hx^2\right]$$

and so

$$\left[h, 3bh^2z + 2fAh^2x - 3Ph^2x - \frac{3}{2}\alpha hx^2\right] = 0.$$

By Lemma 11 there exists $l \in \mathbb{C}$ such that

$$3bh^2z + 2fh^2x - 3Ph^2x - \frac{3}{2}\alpha hx^2 = lh^3.$$

Since $h^2 | 3bh^2z + 2fAh^2x - 3Ph^2x$ and $\alpha \neq 0$, we see that $h|x$. But this means that $h = \mu x$ for some $\mu \in \mathbb{C}^*$ (remember that $\alpha h^6 \neq 0$). But this contradicts $h = \frac{1}{M}\left[z - \left(E - \frac{3}{4}bA + \frac{3}{4}\alpha f\right)x\right]$.

Now, assume that $F$ and $G$ satisfy Lemma 27(1). Then

$$[F_3, z] = \left[bh^3, M\overline{F}_1 + Nh\right] = 3bMh^2\left[h, \overline{F}_1\right], \tag{121}$$

$$\begin{aligned} [F_2, G_2] &= \left[h\overline{F}_1, \widetilde{B}h^2 + C\overline{F}_1^2 + \widetilde{D}h\overline{F}_1\right] = \left[h\overline{F}_1, \widetilde{B}h^2 + C\overline{F}_1^2\right]\\ &= h\left[\overline{F}_1, \widetilde{B}h^2 + C\overline{F}_1^2\right] + \overline{F}_1\left[h, \widetilde{B}h^2 + C\overline{F}_1^2\right]\\ &= 2\widetilde{B}h^2\left[\overline{F}_1, h\right] + 2C\overline{F}_1^2\left[h, \overline{F}_1\right] \end{aligned} \tag{122}$$

and

$$\begin{aligned} [x, G_3] &= \left[R\overline{F}_1 + Sh, Kh^3 + \widetilde{L}h^2\overline{F}_1\right]\\ &= R\left[\overline{F}_1, Kh^3 + \widetilde{L}h^2\overline{F}_1\right] + S\left[h, Kh^3 + \widetilde{L}h^2\overline{F}_1\right]\\ &= 3KRh^2\left[\overline{F}_1, h\right] + 2R\widetilde{L}h\overline{F}_1\left[\overline{F}_1, h\right] + S\widetilde{L}h^2\left[h, \overline{F}_1\right]. \end{aligned} \tag{123}$$

By (120)-(123),

$$\left[\left(3bM - 2\widetilde{B} - 3KR + S\widetilde{L}\right)h^2 - 2R\widetilde{L}h\overline{F}_1 + 2C\overline{F}_1^2\right] \cdot \left[h, \overline{F}_1\right] = 0.$$

By Lemma 27(1), $h$ and $\overline{F}_1$ are algebraically independent. Thus $\left[h, \overline{F}_1\right] \neq 0$. Since also $C = \frac{3}{8}\alpha \neq 0$, we see that $\left(3bM - 2\widetilde{B} - 3KR + S\widetilde{L}\right) h^2 - 2R\widetilde{L}h\overline{F}_1 + 2C\overline{F}_1^2 \neq 0$, a contradiction. □

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